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Topological Quantum Field Theory on non-Abelian gerbes

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Abstract

The infinitesimal symmetries of a fully decomposed non-Abelian gerbe can be generated in terms of a nilpotent BRST operator, which is here constructed. The appearing fields find a natural interpretation in terms of the universal gerbe, a generalisation of the universal bundle. We comment on the construction of observables in the arising Topological Quantum Field Theory. It is also shown how the BRST operator and the trace part of a suitably truncated set of fields on the non-Abelian gerbe reduce directly to the coboundary operator and the pertinent cochains of the underlying Čech–de Rham complex.

Keywords: Non-Abelian gerbe; BRST symmetry; Universal gerbe

1. Introduction

The natural generalisation of a principal bundle is a non-Abelian gerbe [1]. There are different ways of defining such an object. In this paper we shall make use of the very general approach of Ref. [2] based on category theory [3]. Other approaches include [4,5]. Generalisations of Yang–Mills theory have been discussed generally e.g. in [6,7]. Non-Abelian two-forms and their uses in loop space have been approached e.g. in [8–10]. An example in Supergravity can be found in [11]. Gerbes have appeared in String Theory e.g. in [12–17], in M-theory in [18–20], and in a slightly different incarnation in Quantum Field Theory in e.g. [21].

The aim of the paper is to define a nilpotent BRST operator on non-Abelian gerbes and to develop methods for using non-Abelian gerbes in path integral quantisation of geometrically defined field theories. From the String Theory point of view the emphasis is on the discussion of semiclassical backgrounds rather than defining stringy holonomies. As is usual in Physics, we study the geometric object through fields that live on it: in the case of a principal bundle we might equip it with a connection so that representatives of its characteristic classes can be studied conveniently. In the case of a non-Abelian gerbe we need much more data. The requisite objects were found in [2] making use of recently developed methods in combinatorial differential geometry [22].

We shall first cast some of the results of [2] in a form which is perhaps more immediately applicable in physical problems. In particular, we define the BRST operator of a non-Abelian gerbe as a nilpotent Grassmann odd operator that generates its infinitesimal symmetries. Instrumental to the construction is the universal gerbe which arises as

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a generalisation of the universal bundle [23]. The BRST operator can be discovered as a covariant derivative on it. Universal gerbes in a slightly different context were discussed also in [24]. As the BRST operator implements a shift symmetry, it leads to a topological theory, akin to the standard Topological Yang–Mills theory [25,26]. Topological Quantum Field Theory with Abelian gerbes was also discussed more abstractly for instance in [27].

The method of choice for describing the structure of a fully decomposed gerbe is combinatorial differential geometry [22]. This is perhaps unfamiliar in the physics literature; a brief and informal review of the basic tools is included in Section 2. Most of the discussion is on the algebraic level, and we use heuristic methods, such as path integrals, only to motivate definitions. To give a flavour of the novelties, the requisite tool-kit contains, among other things, three different "derivatives":

- the classical Lie-algebra valued covariant exterior derivative d_A that acts on Lie-algebra valued differential forms;
- the combinatorial differentials $\delta_m^{(n)}$ that act on group-valued differential forms; and
- the generalisation of the Čech coboundary operator, ∂_λ.

These differentials depend characteristically on different data: here A is a locally defined Lie-algebra valued one-form, m a combinatorial Lie-group valued one-form, and λ, in the simplest case, an element of the automorphism group of the underlying Lie-group.

In Section 3 the infinitesimal symmetries of a fully decomposed gerbe are found, and a provisional BRST operator Q is written down. This provisional operator fails to be nilpotent, however, when operated on one of the ghost fields.

Section 4 is a review of the universal bundle, and its uses for defining observables in Donaldson–Witten theory. In Section 5 we generalise the construction for the non-Abelian gerbe, and write down a fully nilpotent BRST operator q for the associated Topological Quantum Field Theory. In Section 6 we change the grading of the BRST operator, and show that the new operator $\bar{\mathrm{q}}$ reduces to Q on-shell.

In Section 7 we discuss defining BRST-closed functionals. Due to the intricate structure of the field content, the simplest such functionals are also BRST-exact and therefore trivial in BRST cohomology. The complications that arise in defining invariant polynomials are intimately related to the rôle played by the outer automorphisms of the underlying gauge symmetry group. It remains an interesting problem to calculate the cohomology of the BRST operator. We finish by showing how the trace part of the present construction produces the Abelian gerbe [28], and its symmetries.

2. Structure of a non-Abelian gerbe

In this section we set the stage for later constructions. We shall first recall the basic group theoretical structures behind a non-Abelian gerbe [29,3], then review aspects of differential calculus with group-valued forms [22], and finally summarise in Section 2.3 the differential geometry of a fully decomposed gerbe [2].

2.1. Cohomology of a gerbe

We give here a brief account of cohomology of gerbes. Note that this cohomology is not related *a priori* to the cohomology of the BRST operator that is the main topic of this paper.

In the Abelian case, the cohomology class of a gerbe with connection and curving is a class in Čech–de Rham cohomology [28]. This generalises readily to arbitrary degree. There is a well-defined characteristic class, which for an n-gerbe on a manifold X is an element of $H^{n+2}(X, \mathbb{R})$.

In the non-Abelian case the situation is directly analogous only at degree one: the cohomology class of a principal G-bundle – seen as a zero-gerbe – is an element of $H^1(X, G)$. The definition makes sense, as the cocycle condition $\lambda_{ij}\lambda_{jk}\lambda_{ki} = \mathbf{1}_i$ is invariant under redefinitions of the local frame $\lambda_{ij} \longrightarrow h_i\lambda_{ij}h_j^{-1}$. When the principal bundle is equipped with a connection, characteristic classes can be defined as elements of $H^*(X, \mathbb{R})$ e.g. in terms of invariant polynomials of the curvature of the connection.

For a one-gerbe, the cocycle condition involves both the automorphism-valued transition function $\lambda_{ij} \in \mathrm{Hom}(G_j, G_i)$ and the group-valued generalisation $g_{ijk} \in G_i$ of the Abelian Čech-cocycle

$$\lambda_{ij}(g_{jkl})g_{ijl} = g_{ijk}g_{ikl} \tag{1}$$

$$\iota_{g_{ijk}}\lambda_{ik} = \lambda_{ij}\lambda_{jk}. \tag{2}$$

The groups G_i could be the same group on each chart $\mathcal{U}_i$, in which case the structure is called a G-gerbe. In what follows we concentrate in the interest of notational simplicity on this case although the analysis goes directly over to the general $\{G_i\}$-gerbe. In any case, the automorphisms λ_{ij} are required to be invertible.

The cohomology of a non-Abelian G-gerbe involves, therefore, both the group G and the automorphisms $\mathrm{Aut}\, G$. Inner automorphisms $\mathrm{Int}\, G$ are given by conjugation with a group element; outer automorphisms are the rest

$$\mathrm{Out}\, G := \mathrm{Aut}\, G/\mathrm{Int}\, G. \tag{3}$$

For a connected, simply connected simple Lie-group G, $\mathrm{Out}\, G$ is given by the symmetries of the Dynkin diagram. Together with the centre of the group $\mathrm{Z}G$ all these groups fit in the exact sequence

$$1 \longrightarrow \mathrm{Z}G \longrightarrow G \xrightarrow{\iota} \mathrm{Aut}\, G \xrightarrow{\sigma} \mathrm{Out}\, G \longrightarrow 1, \tag{4}$$

and in the commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathrm{Z}G & \longrightarrow & G & \longrightarrow & \mathrm{Int}\, G & \longrightarrow & 1 \\
 & & {\scriptstyle\iota}\downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & 1 & \longrightarrow & \mathrm{Aut}\, G & = & \mathrm{Aut}\, G & \longrightarrow & 1
\end{array} . \tag{5}$$

It turns out to be useful to look upon this as a sequence of complexes $(\mathrm{Z}G \longrightarrow 1)$, $(G \xrightarrow{\iota} \mathrm{Aut}\, G)$, and $(\mathrm{Int}\, G \longrightarrow \mathrm{Aut}\, G)$. The last column is by (3) equivalent to $\mathrm{Out}\, G$, and the essence of these diagrams can be boiled down to the "distinguished triangle" [3]

$$\mathrm{Z}G[1] \longrightarrow (G \xrightarrow{\iota} \mathrm{Aut}\, G) \longrightarrow \mathrm{Out}\, G, \tag{6}$$

where "[1]" indicates the shift in degree.

More technically this can be summarised by saying that the cohomology class (λ_{ij}, g_{ijk}) of a non-Abelian G-gerbe is valued in the crossed module $G \xrightarrow{\iota} \mathrm{Aut}\, G$, denoted here with $\mathbf{G}$. The group $H^1(X, \mathbf{G})$ of equivalence classes of such gerbes fits in the exact sequence [29]

$$H^0(X, \mathrm{Out}\, G) \longrightarrow H^2(X, \mathrm{Z}G) \longrightarrow H^1(X, \mathbf{G}) \longrightarrow \mathrm{Tors}\,(\mathrm{Out}\, G), \tag{7}$$

where $\mathrm{Tors}\, H$ refers to isomorphism classes of principal H-bundles. The shift in the degree is due to the fact that $\mathbf{G}$ is a complex. Therefore, if there are no outer automorphisms, the gerbe $\mathbf{G}$ is cohomologically an Abelian $\mathrm{Z}G$-gerbe.

2.2. *Group-valued differential forms*

In this section we review informally basic techniques for calculating with group-valued differential forms needed later in the paper. For a more systematic account, see [22,2].

Let $\underline{\Omega}^*(X, G)$ denote the sheaf[1] of group G-valued local differential forms on X relative to a fixed cover $\{\mathcal{U}_i\}$ of X. To be quite concrete, a typical element in it is a rank-n combinatorial differential form $\alpha_{i_1 \cdots i_k}$ in $\Omega^n(\mathcal{U}_{i_1 \cdots i_k}, G_{i_1})$ defined on the k-fold intersection $\mathcal{U}_{i_1 \cdots i_k}$ with coefficients in the local group G_{i_1}. This generalises the Čech–de Rham complex in a natural way. In what follows there is no need to indicate explicitly what the local group G_{i_1} is, because it is implicit in the first index of the intersection, i_1: in lieu of G_{i_1} we write simply G.

Let $g, h \in \underline{\Omega}^*(X, G)$ and $\mu, \nu, \lambda \in \underline{\Omega}^*(X, \mathrm{Aut}\,(G))$. The group commutator for G and $\mathrm{Aut}\, G$-valued fields is defined as

$$[g, h] := ghg^{-1}h^{-1}, \tag{8}$$

and similarly for $\mathrm{Aut}\, G$. When the degree of both fields is positive, the group commutator reduces (on the level of one-jets cf. [22]) to the classical Lie-bracket

$$[g, h] = gh - hg. \tag{9}$$

[1] For a precise definition see [22].

This is simply because on the level of one-jets $g = 1 + x + \mathcal{O}^2$ and $h = 1 + y + \mathcal{O}^2$ so that $[g, h] = xy - yx + \mathcal{O}^3$. In combinatorial differential calculus it is unnecessary to distinguish notationally between, for instance, g and x, and we shall indeed change the point of view from the group level to the algebra level as suitable.

The action of elements μ of $\underline{\Omega}^*(X, \mathrm{Aut}\,(G))$ on those of $g \in \underline{\Omega}^*(X, G)$ is denoted as $\mu(g)$. Their commutator is defined as

$$[\mu, g] := \mu(g)g^{-1} \qquad \in \underline{\Omega}^*(X, G), \tag{10}$$

which on the one-jet level $\underline{\Omega}^*(X, \mathrm{Lie}\,(G))$ reduces for positive degree fields to the classical (graded) Lie bracket. One can also define the bracket

$$[g, \mu] := -[\mu, g], \tag{11}$$

which is still group-valued. When the degree of both fields is again positive, it too reduces to classical Lie brackets.

All of these classical Lie-brackets, be their arguments A, B, C Lie G or Lie Aut G-valued differential forms, obey the usual graded classical Jacobi identity,

$$(-)^{|A||C|}[A, [B, C]] + (-)^{|C||B|}[C, [A, B]] + (-)^{|B||A|}[B, [C, A]] = 0 \tag{12}$$

as well as the Leibnitz rule

$$\mathrm{d}_m[A, B] = [\mathrm{d}_m A, B] + (-)^{|A|}[A, \mathrm{d}_m B], \tag{13}$$

where $|A|$ is the degree of A etc., and d_m is a classical covariant derivative

$$\mathrm{d}_m := \mathrm{d} + [m, \ \cdot\ ]. \tag{14}$$

Definiton 2.1. The adjoint action $\iota : G \longrightarrow \mathrm{Aut}\,(G)$ is

$$\iota_g(h) := ghg^{-1}. \tag{15}$$

The automorphism group Aut G acts on itself by

$$^{\lambda}\nu := \lambda\nu\lambda^{-1}. \tag{16}$$

Lemma 1. *Adjoint action ι_g by a group element $g \in G$ enjoys the properties*

$$[g, h] = [\iota_g, h] \tag{17}$$

$$^{\lambda}\iota_g = \iota_{\lambda(g)} \tag{18}$$

$$[\lambda, \iota_g] = \iota_{[\lambda, g]} \tag{19}$$

where $\lambda \in$ Aut G and $h \in G$.

Proof. First, definition of the commutators $[g, h] = ghg^{-1}h^{-1} = \iota_g(h)\,h^{-1} = [\iota_g, h]$; second, elements of Aut (G) are homomorphisms and $^{\lambda}\mu = \lambda\mu\lambda^{-1}$; third, $[\lambda, \iota_g](h) = \lambda(g)\,g^{-1}hg\,\lambda(g)^{-1} = [\lambda, g]h[\lambda, g]^{-1}$. □

The combinatorial covariant derivatives

$$\delta_m^{(n)} : \underline{\Omega}^n(X, G) \longrightarrow \underline{\Omega}^{n+1}(X, G), \qquad n \geq 0 \tag{20}$$

reduce to the classical covariant derivatives

$$\delta_m^{(n)}\omega = \mathrm{d}\omega + [m, \omega] = \mathrm{d}_m\omega, \qquad n \geq 2 \tag{21}$$

$$\delta_m^{(1)}\omega = \mathrm{d}_m\omega + \frac{1}{2}[\omega, \omega], \tag{22}$$

except for $n = 0$. Note that $\delta_m^{(n+1)}\delta_m^{(n)}\omega = [\kappa(m), \omega]$ for $n = 0$ and $n \geq 2$ with

$$\kappa(m) = \mathrm{d}m + \frac{1}{2}[m, m], \tag{23}$$

Table 1
The local fields on a non-Abelian gerbe

	0-form	1-form	2-form	3-form
G	g_{ijk}	γ_{ij}	$B_i,\ \delta_{ij}$	ω_i
Aut(G)	λ_{ij}	m_i	ν_i	

whereas for $n = 1$ we have

$$\delta_m^{(2)}\delta_m^{(1)}\omega = [\kappa(m) + \mathrm{d}_m\omega, \omega]. \tag{24}$$

Also

$$\delta_m^{(1)}(-\omega) = -\delta_m^{(1)}\omega + [\omega, \omega]. \tag{25}$$

There is an alternative set of differentials

$$\tilde{\delta}_m^{(n)}\omega := -\delta_m^{(n)}(-\omega) \tag{26}$$

which of course coincides with the $\delta_m^{(n)}$ for $n \geq 2$. The analogue of the Čech differential is

$$\partial_\lambda\omega_{ij} := \omega_{ij} + \lambda_{ij}(\omega_{jk}) + \lambda_{ij}\lambda_{jk}(\omega_{ki}). \tag{27}$$

2.3. *A fully decomposed gerbe*

The differential geometry of a non-Abelian gerbe [2] involves the fields summarised in Table 1. The *cocycle data* (λ_{ij}, g_{ijk}) satisfies

$$\lambda_{ij}(g_{jkl})g_{ijl} = g_{ijk}g_{ikl} \tag{28}$$

$$\iota_{g_{ijk}}\lambda_{ik} = \lambda_{ij}\lambda_{jk}. \tag{29}$$

We add to this the connection m_i and the two-form B_i and define

$$\omega_i := \mathrm{d}_{m_i}(B_i) \tag{30}$$

$$\nu_i := \kappa(m_i) - \iota_{B_i}. \tag{31}$$

The covariant derivative is the standard $\mathrm{d}_m B := \mathrm{d}B + [m, B]$ with curvature $\kappa(m) := \mathrm{d}m + \frac{1}{2}[m, m]$. It is compatible with the inner action ι in the sense that $\iota_{d_m(B)} = d_m\iota_B$. Note also that we will use this definition inherited from Lie-algebra valued differential forms everywhere, including in the case of one-forms where Refs. [22,2] use[2] $\delta_m^1(\gamma) := \mathrm{d}_m\gamma + \frac{1}{2}[\gamma, \gamma]_m$.

To relate these fields m_i and B_i on different charts, we need γ_{ij} and δ_{ij} such that

$${}^{\lambda_{ij}*}m_j - m_i = -\iota_{\gamma_{ij}} \tag{32}$$

$$\lambda_{ij}(B_j) - B_i = \delta_{ij} - \mathrm{d}_{m_i}(\gamma_{ij}) + \frac{1}{2}[\gamma_{ij}, \gamma_{ij}]_{m_i}. \tag{33}$$

The star in the action of λ_{ij} here refers to the fact that m_i transforms as a gauge field ${}^{\lambda_{ij}*}m_j := {}^{\lambda_{ij}}m_j + \lambda_{ij}\mathrm{d}\lambda_{ij}{}^{-1}$. We can view (33) as a definition of the G-valued two-form δ_{ij}

$$\delta_{ij} := \lambda_{ij}(B_j) - B_i + \mathrm{d}_{m_i}(\gamma_{ij}) - \frac{1}{2}[\gamma_{ij}, \gamma_{ij}]_{m_i}, \tag{34}$$

[2] The factor of 1/2 is crucial: this definition together with the result (6.1.19) of [2]$\delta^1(-\gamma_{ij}) = -\delta^1(\gamma_{ij}) + [\gamma_{ij}, \gamma_{ij}]$ can be used to turn the cocycle condition (6.1.18) of [2] $\delta^1(-\gamma_{ij}) = -\mathrm{d}_{m_i}(\gamma_{ij}) + (1 - \frac{1}{2})[\gamma_{ij}, \gamma_{ij}]_{m_i}$ into the form required here in Eq. (34).

whereas (32) determines only the inner action of γ_{ij}. Note that the twisted commutators $[\gamma_{ij}, \gamma_{ij}]_{m_i}$ are actually independent of the twisting one-form m_i, cf. (A.1.23) of [2], so we can treat them safely as standard untwisted commutators. This leads [2] to the cocycle conditions

$$\partial_{\lambda_{ij}}(\gamma_{ij}) = \tilde{\mathrm{d}}_{m_i}(g_{ijk}) \tag{35}$$

$$\partial_{\lambda_{ij}}(\delta_{ij}) = [\nu_i, g_{ijk}]. \tag{36}$$

The covariant derivative of group-valued functions $\tilde{\mathrm{d}}_{m_i}(g_{ijk})$ cannot be easily represented in terms of algebra-valued expressions.[3]

We call the triple (m_i, γ_{ij}, B_i) *connection data*. Here δ_{ij} and ν_i belong to the *curvature triple* $(\nu_i, \delta_{ij}, \omega_i)$. The cocycle conditions and the transformation properties of the curvature triple are in addition to the above equations

$$\iota_{\omega_i} = -\mathrm{d}_{m_i}(\nu_i) \tag{37}$$

$$\mathrm{d}_{m_i}(\omega_i) = [\nu_i, B_i] \tag{38}$$

$${}^{\lambda_{ij}}\nu_j - \nu_i = -\iota_{\delta_{ij}} \tag{39}$$

$$\lambda_{ij}(\omega_j) - \omega_i = \mathrm{d}_{m_i}(\delta_{ij}) + [\gamma_{ij}, \nu_i] - [\gamma_{ij}, \delta_{ij}]. \tag{40}$$

One of the consequences of these cocycle conditions is the fact that if the *fake curvature* ν_i vanishes, then by (37) and (39) the rest of the curvature data are Abelian.

2.4. Exact symmetries

The freedom to choose a basis in each chart $\mathcal{U}_i$ of the manifold gives rise to the *local gauge symmetry*: given local functions $h_i \in \Omega^0(\mathcal{U}_i, \mathrm{Aut}\,(G_i))$ we may change the basis by

$$m_i \longrightarrow {}^{h_i *}m_i \qquad := h_i \mathrm{d}_{m_i}(h_i^{-1}) \tag{41}$$

$$\gamma_{ij} \longrightarrow h_i(\gamma_{ij}) \tag{42}$$

$$B_i \longrightarrow h_i(B_i) \tag{43}$$

and so on. Under these symmetries the cocycle conditions transform obviously covariantly. Connection data deserves its name because it can be shifted by *affine data* $(\pi_i, \eta_{ij}, \alpha_i, E_i)$ that satisfy the cocycle conditions [2]

$${}^{\lambda_{ij}}\pi_j - \pi_i = -\iota_{\eta_{ij}} \tag{44}$$

$$\partial_{\lambda_{ij}}(\eta_{ij}) = [\pi_i, g_{ijk}]. \tag{45}$$

The transformation rules of the connection data are

$$m_i' - m_i = \pi_i + \iota_{E_i} \tag{46}$$

$$\gamma_{ij}' - \gamma_{ij} = \eta_{ij} - \lambda_{ij}(E_j) + E_i \tag{47}$$

$$B_i' - B_i = \alpha_i + \kappa(E_i) + [m_i, E_i] + [\pi_i, E_i]. \tag{48}$$

This induces the following symmetry on the curvature triple:

$$\nu_i' - \nu_i = \kappa(\pi_i) + [m_i, \pi_i] - \iota_{\alpha_i} \tag{49}$$

$$\begin{aligned}\delta_{ij}' - \delta_{ij} \ &= \ \lambda_{ij}(\alpha_j) - \alpha_i + \mathrm{d}_{m_i}(\eta_{ij}) - [\eta_{ij}, \eta_{ij}]_{m_i} \\ &\quad + [\pi_i, \eta_{ij}]_{m_i} - [\gamma_{ij}, \eta_{ij}]_{m_i} + [\gamma_{ij}, \pi_i]_{m_i}\end{aligned} \tag{50}$$

$$\begin{aligned}\omega_i' - \omega_i \ &= \ \mathrm{d}\alpha_i + [m_i, \alpha_i] + [\pi_i, B_i + \alpha_i] \\ &\quad - [\alpha_i, E_i] + [\nu_i + \kappa(\pi_i) + [m_i, \pi_i], E_i].\end{aligned} \tag{51}$$

[3] The notation of [2] used $\tilde{\delta}^0(g_{ijk}) = {}^{g}\delta^0_{m_i}(g_{ijk}) = \tilde{\mathrm{d}}_{m_i}(g_{ijk})$.

We call this symmetry the *affine gauge symmetry*. The affine data are themselves subject to the symmetry

$$\pi' - \pi = \iota_{\rho_i} \tag{52}$$

$$\eta'_{ij} - \eta_{ij} = \rho_i - \lambda_{ij}(\rho_i) \tag{53}$$

$$E'_i - E_i = -\rho_i \tag{54}$$

$$\alpha'_i - \alpha_i = \kappa(\rho_i) + [m_i, \rho_i] + [\pi_i, \rho_i]. \tag{55}$$

We call this redundancy the *reduced gauge symmetry*.

3. Infinitesimal symmetries

A BRST operator is a nilpotent (of order two) differential on the field space.[4] In order to construct such an operator, we must be able to model differentials of physical fields consistently. We do this formally using the Grassmann algebra $\mathbb{G}$ of anticommuting real numbers. The infinitesimal fields are called ghosts in the physics literature, as they decouple from physical amplitudes. The requirement for nilpotency of the BRST operator may require introducing differentials for ghost fields themselves as well; these objects are called ghost-for-ghost fields. Ghosts-for-ghosts can be thought of as two-forms in the field space. All the emerging fields are graded in terms of the ghost number, and there can, in principle, be an infinite tower of them, though we shall here have to advance up to ghost number three only.

As a field with positive ghost number is an infinitesimal, it gets also at form degree zero its values in the Lie-algebras $\mathrm{Lie}\, G$ and $\mathrm{Lie}\, G \otimes \mathbb{G}$ rather than the respective groups. Indeed, a typical field of odd ghost number is a differential form in $\underline{\Omega}^n(X, \mathrm{Lie}\, G \otimes \mathbb{G})$ for some $n > 0$; for even positive ghost number they are classical differential forms in $\underline{\Omega}^*(X, \mathrm{Lie}\, G)$. This potential discrepancy with combinatorial differential forms will be explained and put in context in Section 5.

In this section we shall begin by writing down a BRST operator "s" that generates infinitesimal versions of the gauge transformations of the last section. Reducibility and nilpotency considerations force us to amend the derivative s to $\mathrm{Q} = \mathrm{s} + \delta + \tilde{\delta}$. The BRST operator we obtain in this way is nilpotent on connection data, but fails to be nilpotent on one of the ghost fields.

3.1. Infinitesimal transformations

The derivative "s" arises from infinitesimal displacements generated by local gauge transformations h_i and the symmetries of the gerbe in Section 2.4. For the finite local gauge transformation $h_i \in \Omega^0(\mathcal{U}_i, \mathrm{Aut}\, G)$ corresponds the infinitesimal, Grassmann-valued ghost field $c_i \in \Omega^0(\mathcal{U}_i, \mathrm{Lie}\,\mathrm{Aut}\, G \otimes \mathbb{G})$. Similarly, the affine data $(\pi_i, \eta_{ij}, \alpha_i, E_i)$ of Section 2.4 are all Grassmann-valued ghost fields in this section. We may now write down the local gauge and affine transformations *in infinitesimal form* for the gauge fields

$$\mathrm{s} m_i = \pi_i + i_{E_i} - \mathrm{d}_{m_i} c_i \tag{56}$$

$$\mathrm{s} \gamma_{ij} = \eta_{ij} - \lambda_{ij}(E_j) + E_i + [c_i, \gamma_{ij}] \tag{57}$$

$$\mathrm{s} B_i = \alpha_i + \mathrm{d}_{m_i}(E_i) + [c_i, B_i] \tag{58}$$

$$\mathrm{s} \alpha_i = -[\pi_i, E_i] + [c_i, \alpha_i] \tag{59}$$

$$\mathrm{s} c_i = \frac{1}{2}[c_i, c_i]. \tag{60}$$

Other fields x transform according to the standard rule

$$\mathrm{s} x = [c_i, x]. \tag{61}$$

As the cocycle data remain constant under the symmetries of the gerbe we set (cf. Section 4.3)

$$\mathrm{s} \lambda_{ij} = 0 \tag{62}$$

$$\mathrm{s} g_{ijk} = 0. \tag{63}$$

[4] See [30] for a thorough treatment. We use here the concept of "field space" heuristically; in Section 5 we present a more detailed description of what we mean by it.

The connection data have ghost number zero, and all the transformation parameters above $c_i, \pi_i, \eta_{ij}, \alpha_i, E_i$ have ghost number one. The derivative s raises ghost number by one. Ghost number grading is independent of form degree grading. In this section, a field of form degree p and ghost number q can be thought of as a real number-valued differential form of degree p when q is *even*. When q is *odd*, the components of the differential form are elements of the Grassmann algebra $\mathbb{G}$. By multiplying two such objects of grading (p, q) and (p', q') we get an object of grading $(p + p', q + q')$. These two bigraded objects are mutually odd precisely when $pp' + qq'$ is odd, otherwise even.[5] The brackets [,] can be graded so that the pertinent graded Jacobi identity applies.

The differences to the original transformations in Section 2.3 are the following:

- We have discarded terms that are of higher order than linear in affine data, namely $[E_i, E_i]/2 + [\pi_i, E_i]$ in $\mathrm{s}B_i$ (48), and $[\rho_i, \rho_i]/2 + [\pi_i, \rho_i]$ in $\mathrm{s}\alpha_i$ (55).
- We have added the extra term $-[\pi_i, E_i]$ in (59). Note that the same term had to be struck off from (48). Also, this term vanishes at the equivariant $c_i = 0$ fixed point locus of the full nilpotent BRST operator, cf. Section 6.3.
- Commutators between two affine-fields-turned-ghosts have been changed to anticommutators, and vice versa.

The justification for these differences is the fact that s does still generate symmetries of the underlying gerbe, though infinitesimal.

The fact that $\mathrm{s}x$ is infinitesimal of order one means that we can extend the action of s to any functional composed of fields whose BRST transformation under s we have defined. s is therefore a graded odd derivation, and raises ghost number by one. Most importantly, it is nilpotent

$$\mathrm{s}^2 = 0. \tag{64}$$

3.2. Reducibility

All of the fields π_i, E_i, and c_i describe shifts in m_i in different ways. Given a specific, fixed shift $m_i' - m_i$ there is latitude in how it is written down in terms of π_i, E_i, and c_i. In Section 2.4 the latitude in the choice of (π_i, E_i) was parametrised in terms of ρ_i. Taking also c_i into account we need two more ghost-for-ghosts $\varphi_i \in \Omega^0(\mathcal{U}_i, \mathrm{Lie\,Aut}\,G)$ and $\phi_i \in \Omega^0(\mathcal{U}_i, \mathrm{Lie}\,G)$.

The new fields force us to amend the BRST differential $\mathrm{s} \longrightarrow \mathrm{s} + \delta$. The new piece, δ, is an odd graded derivation of ghost number one, as was s. The nontrivial action of δ is

$$\delta\pi_i = \mathrm{d}_{m_i}\varphi_i + \iota_{\rho_i} \tag{65}$$

$$\delta E_i = \mathrm{d}_{m_i}\phi_i - \rho_i \tag{66}$$

$$\delta c_i = \varphi_i + \iota_{\phi_i} \tag{67}$$

$$\delta\alpha_i = \mathrm{d}_{m_i}\rho_i + [B_i, \varphi_i] + [\phi_i, \nu_i] \tag{68}$$

$$\delta\eta_{ij} = \rho_i - d_{m_i}\phi_i - \lambda_{ij}\left(\rho_j - \mathrm{d}_{m_j}\phi_j\right) + [\gamma_{ij}, \phi_i] + [\gamma_{ij}, \varphi_i]. \tag{69}$$

These transformations are chosen so that $\delta \mathrm{s} m_i = \delta \mathrm{s}\gamma_{ij} = \delta \mathrm{s} B_i = 0$. As the action of δ on other fields is trivial, δ is nilpotent $\delta^2 = 0$. Note that $\mathrm{s} + \delta$ is not nilpotent, but, for instance,

$$(\mathrm{s} + \delta)^2 E_i = [\varphi_i, E_i] + [\pi_i, \phi_i]. \tag{70}$$

This non-nilpotency can be remedied partially by taking into account that there is a further latitude in defining the $\rho_i, \phi_i, \varphi_i$ system. This latitude has to be parametrised with the ghost number-three field $\sigma_i \in \Omega^0(\mathcal{U}_i, \mathrm{Lie}\,G \otimes \mathbb{G})$. This gives rise to the transformations

$$\bar{\delta}\varphi_i = -\iota_{\sigma_i} \tag{71}$$

$$\bar{\delta}\phi_i = \sigma_i \tag{72}$$

$$\bar{\delta}\rho_i = \mathrm{d}_{m_i}\sigma_i + [\varphi_i, E_i] - [\phi_i, \pi_i] \tag{73}$$

$$\bar{\delta}\sigma_i = [\phi_i, \varphi_i]. \tag{74}$$

The construction is such that $\bar{\delta}\delta(\pi, E, c) = 0$. Again $\bar{\delta}$ annihilates all other fields so that $\bar{\delta}^2 = 0$.

[5] There is another way of doing this, cf. Section 6.1. This convention is the only one immediately consistent with the infinitesimal symmetries of the gerbe though.

Table 2
Fields and their field strengths

(field, ghost)		Curvature	Domain
(m_i, π_i)	$\longrightarrow$	ν_i	$\mathcal{U}_i$
(γ_{ij}, η_{ij})	$\longrightarrow$	δ_{ij}	$\mathcal{U}_{ij}$
(B_i, α_i)	$\longrightarrow$	ω_i	$\mathcal{U}_i$

Theorem 2. *The operator* $\mathrm{Q} := \mathrm{s} + \delta + \bar{\delta}$ *is an odd derivation of ghost number one. It is nilpotent* $\mathrm{Q}^2 x = 0$ *on all fields x where we have defined it, except on* $x = \eta_{ij}$

$$\mathrm{Q}^2\eta_{ij} = -\left[^{\lambda_{ij}}\left(\varphi_j + \iota_{\phi_j}\right) - (\varphi_i + \iota_{\phi_i}), \quad \lambda_{ij}(E_j)\right] - \left[^{\lambda_{ij}} c_j - c_i, \quad \lambda_{ij}(\rho_j - \mathrm{d}_{m_j}\phi_j)\right]. \tag{75}$$

The operator Q does therefore not quite qualify as a BRST operator. Note that Q is nevertheless nilpotent in particular on the connection data

$$\mathrm{Q}^2 B_i = \mathrm{Q}^2\gamma_{ij} = \mathrm{Q}^2 m_i = 0. \tag{76}$$

Furthermore, the above obstruction to nilpotency vanishes if $\varphi_i + \iota_{\phi_i}$ and c_i extend to global sections.

The resolution to this problem is to introduce new fields a_{ij} and b_{ij} that control the behaviour of c_i and φ_i on double-intersections $\mathcal{U}_{ij}$. In pursuing this, the relationship of the BRST operator to the symmetries of the underlying gerbe becomes slightly obscured. This happens necessarily because the procedure requires essentially replacing, for instance, the term $d_{m_i}\phi_i - \lambda_{ij}(\mathrm{d}_{m_j}\phi_j)$ in (69) by the covariant derivative of one of the new fields $\mathrm{d}_{m_i} b_{ij}$.

Instead of amending Q here further, in Section 5 we will define an operator q which is by construction nilpotent, and that reduces to Q on-shell. This will require a more geometric understanding of the BRST differential at our disposal.

3.3. The curvature triple

To each (field, ghost) pair one may associate a curvature as in Table 2. It can be shown now that to linear order the local, affine, and reduced gauge transformations of the curvature triple $(\nu_i, \delta_{ij}, \omega_i)$ in Section 2.4 arise, as expected, from those of the underlying connection data modulo terms that vanish when the constraints

$$\mathcal{C}^0_{ij} := {}^{\lambda_{ij}*}m_j - m_i + \iota_{\gamma_{ij}} \approx 0 \tag{77}$$

$$\mathcal{C}^0_{ijk} := \partial_{\lambda_{ij}}(\gamma_{ij}) - \tilde{\mathrm{d}}_{m_i}(g_{ijk}) \approx 0 \tag{78}$$

$$\mathcal{B}^1_{ij} := {}^{\lambda_{ij}}\pi_j - \pi_i + \iota_{\eta_{ij}} \approx 0 \tag{79}$$

$$\mathcal{B}^1_{ijk} := \partial_{\lambda_{ij}}(\eta_{ij}) - [\pi_i, g_{ijk}] \approx 0 \tag{80}$$

are imposed. These constraints arose as cocycle conditions in (32), (35), (44) and (45); their rôle is to relate data on different charts to each other.

The cocycle conditions (36) and (37)–(40) for the curvature triple are similarly satisfied up to terms proportional to these constraints. The cocycle conditions (37)–(40) are easy enough to verify using standard de Rham calculus with Lie-algebra valued differential forms. The cocycle condition (36) requires special attention, however, because it involves derivatives of a group-valued local *function*. We explain how this comes about carefully in

Theorem 3. *The cocycle condition*

$$\partial_{\lambda_{ij}}\delta_{ij} \approx [\nu_i, g_{ijk}] \tag{81}$$

arises as a consequence of the cocycle conditions (32) and (35) or, equivalently, as a consequence of the constraints $\mathcal{C}^0_{ij} \approx \mathcal{C}^0_{ijk} \approx 0$.

Proof. One applies $\partial_{\lambda_{ij}}$ on the definition of δ_{ij} in Eq. (34). Note first that the derivative $\partial_{\lambda_{ij}}$ has the essential property

$$\partial_{\lambda_{ij}}\left(\lambda_{ij}(B_j) - B_i\right) = -[\iota_{B_i}, g_{ijk}]. \tag{82}$$

After obvious cancellations this can be seen as follows:

$$\iota_{g_{ijk}}(B_i) - B_i = g_{ijk}B_i g_{ijk}{}^{-1}\, B_i^{-1} = [g_{ijk}, B_i] = -[B_i, g_{ijk}] = -[\iota_{B_i}, g_{ijk}]. \tag{83}$$

Using now repeatedly the constraint $\mathcal{C}_{ij} \approx 0$ we arrive at the expression

$$\partial_{\lambda_{ij}}\delta_{ij} \approx -[\iota_{B_i}, g_{ijk}] + \mathrm{d}_{m_i}\partial_{\lambda_{ij}}\gamma_{ij} - \frac{1}{2}[\partial_{\lambda_{ij}}\gamma_{ij}, \partial_{\lambda_{ij}}\gamma_{ij}]. \tag{84}$$

At this point we have to return to the group-valued differential forms, and the notation of Ref. [2]: the above-appearing expression

$$\mathrm{d}_{m_i}\partial_{\lambda_{ij}}\gamma_{ij} - \frac{1}{2}[\partial_{\lambda_{ij}}\gamma_{ij}, \partial_{\lambda_{ij}}\gamma_{ij}] \tag{85}$$

can then be cast in the form

$$\begin{aligned}
&= \delta^1_{m_i}(\partial_{\lambda_{ij}}\gamma_{ij}) - [\partial_{\lambda_{ij}}\gamma_{ij},\ \partial_{\lambda_{ij}}\gamma_{ij}] && \text{by def. of } \delta^1_m && (86)\\
&= -\delta^1_{m_i}(-\partial_{\lambda_{ij}}\gamma_{ij}) && \text{by Eq. (6.1.19) of [2]} && (87)\\
&\approx -\delta^1_{m_i}(-\tilde{\delta}^0_{m_i} g_{ijk}) && \text{by Eq. (35)} && (88)\\
&= -\delta^1_{m_i}(\delta^0_{m_i}(-g_{ijk})) && \text{by Remark 6.1 of [2]} && (89)\\
&= -[[\kappa(m_i),\ g_{ijk}^{-1}]] && \text{by Eq. (A.1.13) of [2]} && (90)\\
&= +[\kappa(m_i),\ g_{ijk}]. && && (91)
\end{aligned}$$

Combining this with the definition of ν_i concludes the proof. □

4. Topological Yang–Mills theory

Consider isomorphism classes $p \in \mathcal{P}$ of principal bundles L_p with connection and possibly other data on a fixed manifold X. The universal bundle $\mathfrak{P} \longrightarrow X \times \mathcal{P}$ fits in the diagram

$$\begin{tikzcd}
L_p \arrow[r] \arrow[d, "\pi"'] & \mathfrak{P} \arrow[d, "\Pi"] \\
X \times \{p\} \arrow[r, "i"] & X \times \mathcal{P}.
\end{tikzcd} \tag{92}$$

In the case of Topological Yang–Mills Theory [25,31] (cf. [32] for a review) we fix the local transition functions ℓ_{ij} consistently $\ell_{ij}\ell_{jk}\ell_{ki} = \mathbf{1}$ once and for all, and keep free only the local connection one-form $m \in \mathcal{A}$ on the bundle. The arising universal bundle $\mathfrak{P}$ is locally of the form $L_p \times \mathcal{A}$. The gauge equivalence classes of the connections $\mathcal{P} = \mathcal{A}/\mathcal{G}$ do not necessarily form a smooth manifold; the universal bundle, nevertheless, has a smooth base space, which is locally of the form $(L_p \times \mathcal{A})/\mathcal{G}$. If $\mathcal{G}$ acts freely on $\mathcal{A}$, this reduces to $X \times \mathcal{A}/\mathcal{G}$, and we identify $\mathcal{P} = \mathcal{A}/\mathcal{G}$.

As all objects transform in the Yang–Mills case covariantly between charts, there is therefore no need to indicate the local chart, and we will suppress the pertinent indices for a moment. Choosing $(g_{ijk}, \lambda_{ij}) = (\mathbf{1}, \ell_{ij})$ the non-Abelian gerbe collapses now to Topological Yang–Mills theory with only m, π, c, φ active, and other fields set to trivial values.

Given a one-form $c \in T^*\mathcal{A}$, we may construct a covariant exterior derivative on $\mathfrak{P}$

$$D_\mu = \mathrm{d}_m + \mathrm{q}_c, \tag{93}$$

where $\mathrm{q}_c X = \mathrm{q}X + [c, X]$. Here d is the exterior derivative on X and q on $\mathcal{A}/\mathcal{G}$, when the latter makes sense. The curvature can be expanded in terms of the bidegree

$$\begin{aligned}
D_\mu^2 &= \mathcal{F}^{(2,0)} + \mathcal{F}^{(1,1)} + \mathcal{F}^{(0,2)} && (94)\\
&:= \kappa(m) + \pi + \varphi; && (95)
\end{aligned}$$

with the latter line we merely mean that e.g. the field π stands for the (1,1) component of the curvature. These definitions imply then, in fact, together with the standard Bianchi identity $D_\mu \mathcal{F} = 0$, the action of q on various fields

$$\mathrm{q}m = \pi - \mathrm{d}_m c \tag{96}$$

$$\mathrm{q}c = \varphi - \frac{1}{2}[c, c] \tag{97}$$

$$\mathrm{q}\pi = -\mathrm{d}_m \varphi - [c, \pi] \tag{98}$$

$$\mathrm{q}\varphi = -[c, \varphi]. \tag{99}$$

4.1. Observables

The Bianchi identity implies also

$$(\mathrm{d} + \mathrm{q})\mathrm{Tr}\,\mathcal{F}^n = 0. \tag{100}$$

Let us decompose $\mathrm{Tr}\,\mathcal{F}^n = \sum_k W_k$, where k is the form degree on X, and integrate each form over $\gamma \subset X$ of suitable dimension k [25]

$$W_k(\gamma) := \left\langle \int_\gamma W_k \right\rangle. \tag{101}$$

For this to be a good observable γ should not have a boundary, as only then is (101) BRST-closed

$$\mathrm{q}W_k(\gamma) = -\left\langle \int_\gamma \mathrm{d}W_{k-1} \right\rangle = 0. \tag{102}$$

On the other hand, changing γ by a boundary ∂s changes this observable by a BRST-exact term

$$W_k(\partial s) = \left\langle \int_s \mathrm{d}W_k \right\rangle = -\left\langle \mathrm{q} \int_s W_{k-1} \right\rangle = 0, \tag{103}$$

so that the vacuum expectation value $W_k(\gamma)$ remains invariant. The vacuum expectation values $W_k(\gamma)$ depend therefore only on the homology class $[\gamma] \in H_k(X)$. In the case of Topological Yang–Mills, $W_k(\gamma)$ are the Donaldson–Witten invariants [25].

4.2. Curvature

Let us decompose – following Ref. [32] and references therein – the BRST operator into horizontal and vertical parts

$$\mathrm{q} = \mathrm{q}^H + \mathrm{q}^V, \tag{104}$$

where q^V acts along the fibre $\mathcal{G}$, and q^H on the base $\mathcal{A}/\mathcal{G}$.

The vertical derivative generates standard gauge transformations

$$\mathrm{q}^V m = -\mathrm{d}_m c \tag{105}$$

$$\mathrm{q}^V c = -\frac{1}{2}[c, c] \tag{106}$$

$$\mathrm{q}^V x = -[c, x]. \tag{107}$$

It is nilpotent $(\mathrm{q}^V)^2 = 0$, so that its curvature vanishes identically. The horizontal part has curvature

$$(\mathrm{q}^H)^2 m = -\mathrm{d}_m \varphi \tag{108}$$

$$(\mathrm{q}^H)^2 x = [\varphi, x], \tag{109}$$

where x is any other field than m. One may think of q^H as the *covariant exterior derivative* [32] on the bundle $\mathcal{A} \longrightarrow \mathcal{A}/\mathcal{G}$, and of φ as its curvature.

4.3. Ghost number in combinatorial differential geometry

Given a differential form on the universal bundle, one can decompose it locally in terms of differential forms on L_p and those on $\mathcal{A}$. The degree of the former is the de Rham degree, and the degree of the latter the ghost number.

Take the points x, y, ξ that are all infinitesimally close in $L_p \times \mathcal{A}$, such that the projections of x and y onto the second factor are identical, and that the projection of x and ξ onto the first factor are identical. Then the connection $\mu \in \underline{\Omega}^1(\mathfrak{P}, G)$ can be decomposed as

$$\mu(x, y) = m(x, y) \tag{110}$$

$$\mu(x, \xi) = c(x). \tag{111}$$

The BRST transformation q is clearly displacement along $\mathcal{A}$

$$\tilde{\delta}^0_\mu g(x, \xi) = \mu(x, \xi)(g(\xi))g(x)^{-1} \tag{112}$$

$$= c(x)(g(\xi))g(x)^{-1} \tag{113}$$

$$= c(x)(g(\xi))g(\xi)^{-1}\, g(\xi)g(x)^{-1}. \tag{114}$$

The last line is really the covariant derivative "$[c, g] + \mathrm{q}g$". If we drop the Faddeev–Popov ghost setting $c = 0$, we get the covariant exterior derivative on the base space $\mathcal{P}$ discussed above, q^H. This means that objects that remain constant in BRST transformations q, are covariantly constant sections of $\mathcal{A} \longrightarrow \mathcal{P}$. This will have interesting repercussions in Section 7.3.

5. The universal gerbe

We started in Section 2.1 with a gerbe whose cohomology class was given by the cocycle data (λ_{ij}, g_{ijk}) in $\mathbf{G}$. In Section 2.3 we recalled the fields needed to decompose the gerbe fully. Let us denote this set of fields – the connection data etc. – for a fixed gerbe in $\mathbf{G}$ by $\hat{\mathcal{A}}$. This notation is justified, as it is clearly a generalisation of the affine space of connections that appeared in Section 4.

Let us denote the symmetries of the fully decomposed gerbe in a similar vein by $\hat{\mathcal{G}}$. Then picking a specific fully decomposed gerbe $P_{\mathbf{g}}$ provides a representative for the equivalence class $\mathbf{g} \in \mathcal{H} = \hat{\mathcal{A}}/\hat{\mathcal{G}}$. The *universal gerbe* $\mathfrak{G}$ can be constructed formally (as a set) as the disjoint union of all such representatives of all isomorphism classes of fully decomposed gerbes, and fits in a similar diagram to that of the universal bundle (92)

$$\begin{tikzcd} P_{\mathbf{g}} \arrow[r] \arrow[d, "\pi"'] & \mathfrak{G} \arrow[d, "\Pi"] \\ X \times \{g\} \arrow[r, "i"] & X \times \mathcal{H}. \end{tikzcd} \tag{115}$$

Again, we keep the cocycle data (λ_{ij}, g_{ijk}) fixed on a fixed cover $\{\mathcal{U}_i\}$ of X, and let the connection data $(m_i, \gamma_{ij}, B_i) \in \hat{\mathcal{A}}$ vary freely. Isomorphism classes in $\mathcal{H}$ are equivalence classes of elements of $\hat{\mathcal{A}}$ identified by symmetries of a gerbe $\hat{\mathcal{G}}$.

To show that the quotient $\mathfrak{G} \longrightarrow \mathfrak{G}/\hat{\mathcal{G}}$ should actually define a smooth bundle would require careful topologising $\mathfrak{G}$ and studying the action of $\hat{\mathcal{G}}$ on it. As in the case of the universal bundle, the existence of a smooth quotient $\mathcal{H} = \hat{\mathcal{A}}/\hat{\mathcal{G}}$ would specifically require further assumptions on the gauge data, such as restricting to irreducible connections only. In the discussion that follows, we shall nevertheless need only the fact that $\mathcal{H}$ provides a local moduli space for fully decomposed gerbes near a fixed reference gerbe, and is not strictly speaking dependent on whether $\mathfrak{G}$ exists as a universal object or not. The local statement is certainly true, though the stronger assertion seems plausible as well. Note that the universal gerbe in the cohomologically Abelian setting of bundle gerbes was defined rigorously in Ref. [24].

We can think of the universal gerbe $\mathfrak{G}$ also as a stack[6] of local universal bundles $\{\mathfrak{P}_i\}$ on X, and invertible morphisms between them $\lambda_{ij} \in \mathrm{End}(\mathfrak{P}_j, \mathfrak{P}_i)$ with extra structure $\hat{\mathcal{A}}$ and symmetries $\hat{\mathcal{G}}$. The symmetries of the gerbe

[6] Or, more correctly, a stack of categories whose objects are local bundles.

Table 3
Fields and field strengths on the universal gerbe

	Ghost#	0-form	1-form	2-form	3-form
G	0	g_{ijk}	γ_{ij}	$B_i,\ \delta_{ij}$	ω_i
	1	a_{ij}	$E_i,\ \eta_{ij}$	α_i	
	2	$\phi_i,\ b_{ij}$	ρ_i		
	3	σ_i			
Aut(G)	0	λ_{ij}	m_i	ν_i	
	1	c_i	π_i		
	2	φ_i			

$\hat{\mathcal{G}}$ include clearly the structure groups $\mathcal{G}_i$ of the underlying local universal bundles $\mathfrak{P}_i$ in a certain way. A mismatch is bound to arise where two universal bundles overlap; the effects of this can be analysed by investigating the behaviour of the horizontal part of the covariant connection on these bundles in Section 5.1.1.

As in the case of the universal bundle, instead of the underlying gerbe $P_{\mathbf{g}} \longrightarrow X \times \{g\}$ in the equivalence class g, we consider the fully decomposed universal gerbe $\mathfrak{G} \longrightarrow X \times \mathcal{H}$ with connection data (μ_i, V_{ij}, A_i). These fields can be expanded in ghost number

$$\mu_i = m_i + c_i \tag{116}$$

$$V_{ij} = \gamma_{ij} + a_{ij} \tag{117}$$

$$A_i = B_i + E_i + \phi_i, \tag{118}$$

where the lowest components (m_i, γ_{ij}, B_i) are the connection data of the underlying gerbe. The higher components appear in the affine and the gauge transformation data of the underlying gerbe on X; as in the case of the universal bundle, ghost fields find a natural place in the higher components of the connection data. Here only $a_{ij} \in \Omega^0(\mathcal{U}_{ij}, \mathrm{Lie}\, G \otimes \mathbb{G})$ is new in the non-Abelian construction, and in Section 7.3 we shall see that it is actually required for the standard Čech–de Rham Abelian construction.

In what follows, two bigraded fields with bigrading (p, q) resp. (p', q') are mutually odd precisely when both the total gradings $p + q$ and $p' + q'$ are odd. In this way all fields can be treated consistently as differential forms on the universal gerbe, rather than differential forms on the underlying manifold with an additional (ghost number) grading.

The curvatures are defined precisely in the same way as in Section 2.3, though now they can be expanded according to the ghost number of each component

$$F_i \;= \nu_i + \pi_i + \varphi_i \tag{119}$$

$$:= \mathcal{F}_i - \iota_{A_i} \tag{120}$$

$$\Delta_{ij} \;= \delta_{ij} + \eta_{ij} + b_{ij} \tag{121}$$

$$:= \lambda_{ij}(A_j) - A_i + D_{\mu_i} V_{ij} - \frac{1}{2}[V_{ij}, V_{ij}] \tag{122}$$

$$\Omega_i \;= \omega_i + \alpha_i + \rho_i + \sigma_i \tag{123}$$

$$:= D_{\mu_i} A_i. \tag{124}$$

All these fields can be collected in Table 3.

5.1. *The differentials along the universal gerbe*

These definitions determine the curvature triple $(\nu_i, \delta_{ij}, \omega_i)$ in terms of the connection data (m_i, γ_{ij}, B_i), as well as the differentials

$$\mathrm{q} m_i = \pi_i + \iota_{E_i} - \mathrm{d}_{m_i} c_i \tag{125}$$

$$\mathrm{q} c_i = \varphi_i + \iota_{\phi_i} - \frac{1}{2}[c_i, c_i] \tag{126}$$

$$\mathrm{q}_c \gamma_{ij} = \eta_{ij} + E_i - \lambda_{ij}(E_j) - \mathrm{d}_{m_i} a_{ij} + [\gamma_{ij}, a_{ij}] \tag{127}$$

$$\mathrm{q}_c a_{ij} = b_{ij} + \phi_i - \lambda_{ij}(\phi_j) + \frac{1}{2}[a_{ij}, a_{ij}] \tag{128}$$

$$\mathrm{q}_c B_i = \alpha_i - \mathrm{d}_{m_i} E_i \tag{129}$$

$$\mathrm{q}_c E_i = \rho_i - \mathrm{d}_{m_i} \phi_i \tag{130}$$

$$\mathrm{q}_c \phi_i = \sigma_i. \tag{131}$$

The form of the Bianchi identities in terms of the universal connection data is the same as in Section 2.3

$$D_{\mu_i} F_i + \iota_{\Omega_i} = 0 \tag{132}$$

$$D_{\mu_i} \Delta_{ij} + \Omega_i - \lambda_{ij}(\Omega_j) + [\iota_{\Delta_{ij}} - F_i, V_{ij}] = [\mathcal{C}_{ij}, \lambda_{ij}(A_j)] \tag{133}$$

$$D_{\mu_i} \Omega_i - [F_i, A_i] = 0, \tag{134}$$

so that the lowest components in ghost number reproduce precisely the corresponding identities on X. Note that we keep track of the constraint functional

$$\mathcal{C}_{ij} := {}^{\lambda_{ij}*}\mu_j - \mu_i + \iota_{V_{ij}} \tag{135}$$

$$= \mathcal{C}^0_{ij} + \mathcal{C}^1_{ij}, \tag{136}$$

where

$$\mathcal{C}^0_{ij} := {}^{\lambda_{ij}*}m_j - m_i + \iota_{\gamma_{ij}} \tag{137}$$

$$\mathcal{C}^1_{ij} := {}^{\lambda_{ij}}c_j - c_i + \iota_{a_{ij}}. \tag{138}$$

This is because the cocycle equations are needed for an off-shell construction of the nilpotent derivative q. Indeed, the higher components can be used to read off the differentials

$$\mathrm{q}_c \pi_i = -\iota_{\rho_i} - \mathrm{d}_{m_i} \varphi_i \tag{139}$$

$$\mathrm{q}_c \varphi_i = -\iota_{\sigma_i} \tag{140}$$

$$\begin{aligned} \mathrm{q}_c \eta_{ij} = {} & -\mathrm{d}_{m_i} b_{ij} - \rho_i + \lambda_{ij}(\rho_j) + [\pi_i - \iota_{\eta_{ij}}, a_{ij}] + [\varphi_i - \iota_{b_{ij}}, \gamma_{ij}] \\ & - [\mathcal{C}^0_{ij}, \lambda_{ij}(\phi_j)] - [\mathcal{C}^1_{ij}, \lambda_{ij}(E_j)] \end{aligned} \tag{141}$$

$$\mathrm{q}_c b_{ij} = -\sigma_i + \lambda_{ij}(\sigma_j) + [\varphi_i - \iota_{b_{ij}}, a_{ij}] - [\mathcal{C}^1_{ij}, \lambda_{ij}(\phi_j)] \tag{142}$$

$$\mathrm{q}_c \alpha_i = -\mathrm{d}_{m_i} \rho_i + [\nu_i, \phi_i] + [\pi_i, E_i] + [\varphi_i, B_i] \tag{143}$$

$$\mathrm{q}_c \rho_i = -\mathrm{d}_{m_i} \sigma_i + [\pi_i, \phi_i] + [\varphi_i, E_i] \tag{144}$$

$$\mathrm{q}_c \sigma_i = [\varphi_i, \phi_i]. \tag{145}$$

Theorem 4. *The exterior derivative* q *is an odd, identically nilpotent (of order two) differential in the field space.*

Proof. Follows immediately from the definition of q as the exterior derivative on $\mathcal{H}$ from the point of view of the universal gerbe $\mathfrak{G} \longrightarrow X \times \mathcal{H}$. It is instructive to verify this by a direct calculation as well. □

Modulo a few sign differences, which we shall discuss in detail in Section 6, the action of q is on-shell the same as the BRST operator Q in Section 3 and Theorem 2 in particular.

5.1.1. Horizontal derivative

As in Section 4.2, one may again decompose $\mathrm{q} = \mathrm{q}^H + \mathrm{q}^V$, where all c_i dependence is collected in q^V; this makes q^V effectively into a translation along the orbit of local gauge transformations $\mathcal{G}_i$. One may verify that the vertical derivative is still nilpotent $(\mathrm{q}^V)^2 = 0$. The horizontal differential squares to $(\mathrm{q}^H)^2 x = [\varphi_i + \iota_{\phi_i},\ x]$ as expected on all other fields than

$$(\mathrm{q}^H)^2 \eta_{ij} = [\varphi_i + \iota_{\phi_i},\ \eta_{ij}] + [{}^{\lambda_{ij}}(\varphi_j + \iota_{\phi_j}) - (\varphi_i + \iota_{\phi_i}), \lambda_{ij}(E_j)] \tag{146}$$

$$(\mathrm{q}^H)^2 b_{ij} = [\varphi_i + \iota_{\phi_i},\ b_{ij}] + [{}^{\lambda_{ij}}(\varphi_j + \iota_{\phi_j}) - (\varphi_i + \iota_{\phi_i}), \lambda_{ij}(\phi_j)]. \tag{147}$$

The extra piece in (146) is the same that obstructs the nilpotency of Q in Theorem 2.

This calculation shows that we may interpret q^H as the covariant exterior derivative on $\mathfrak{G} \longrightarrow \mathfrak{G}/\mathcal{G}_i$ only where the local curvature $\varphi_i + \iota_{\phi_i}$ extends to a well-defined Lie Aut G-valued section. Outside this domain basic functionals are not necessarily covariantly constant on $\mathfrak{G}/\mathcal{G}_i$. This means effectively that it is not possible to separate the local gauge symmetry part $\mathcal{G}_i$ from the full symmetry group of the gerbe $\hat{\mathcal{G}}$ in any clean way when fields on the double intersections $\mathcal{U}_{ij}$ are taken in general into account.

In Section 8 we shall nevertheless see how the local curvature $\varphi_i + \iota_{\phi_i}$ does extend to a well-defined Lie Aut G-valued section at certain physically relevant configurations, namely fixed point loci of the BRST operator, cf. Section 6.3.

5.2. Constraint algebra

The BRST transformation rule of δ_{ij} can be deduced in two independent ways, on the one hand from the Bianchi identity (133), and on the other by variational calculus from the definition $\delta_{ij} = \delta_{ij}(m_i, \gamma_{ij}, B_i)$. The results must be consistent: this leads us to the observation, as anticipated in Section 3.1, that the structure constants must indeed be held constant in BRST variations $\mathrm{q}\lambda_{ij} = 0$, and that the constraint $\mathcal{C}^1_{ij} \approx 0$ defined in (138) should hold. It follows then $\mathrm{q}g_{ijk} = 0$.

The constraint $\mathcal{C}_{ij} \approx 0$ holds already by definition of the universal gerbe, where the one-form μ_i is a part of the connection data and satisfies therefore the appropriate cocycle conditions. The universal constraints are indeed defined as follows:

Definiton 5.1.

$$\mathcal{C}_{ij} := {}^{\lambda_{ij}*}\mu_j - \mu_i + \iota_{V_{ij}} \tag{148}$$

$$\mathcal{C}_{ijk} := \partial_\lambda V_{ij} + \delta^{(0)}_{\mu_i} g^{-1}_{ijk} \tag{149}$$

$$\mathcal{B}_{ij} := {}^{\lambda_{ij}} F_j - F_i + \iota_{\Delta_{ij}} \tag{150}$$

$$\mathcal{B}_{ijk} := \partial_\lambda \Delta_{ij} - [F_i, g_{ijk}]. \tag{151}$$

The lowest components reproduce

- The constraints $\mathcal{C}^0_{ij}, \mathcal{C}^0_{ijk}, \mathcal{B}^1_{ij}$ and $\mathcal{B}^1_{ijk}$ of the underlying gerbe as in Section 3.3;
- The constraint $\mathcal{C}^1_{ij}$ of (138);
- The cocycle conditions (36) and (39) where the former is identically satisfied $\mathcal{B}^0_{ij} = 0$ and the latter is, by Theorem 3, weakly satisfied $\mathcal{B}^0_{ijk} \approx 0$.

The new constraints are

$$\mathcal{C}^1_{ijk} = \partial_\lambda a_{ij} - [c_i, g_{ijk}] \tag{152}$$

$$\mathcal{B}^2_{ij} = {}^{\lambda_{ij}} \varphi_j - \varphi_i + \iota_{b_{ij}} \tag{153}$$

$$\mathcal{B}^2_{ijk} = \partial_\lambda b_{ij} - [\varphi_i, g_{ijk}]. \tag{154}$$

The reason for imposing these constraints is, again, the geometry of the universal gerbe. On the other hand, there is circumstantial evidence already on the level of the underlying gerbe that they should be imposed: the constraint $\mathcal{B}^2_{ij}$ appears as an obstruction in Theorem 2; the inner parts of the constraints $\mathcal{C}^1_{ijk}$ and $\mathcal{B}^2_{ijk}$ follow as integrality conditions from $\mathcal{C}^1_{ij}$ and $\mathcal{B}^2_{ij}$, respectively.

Whether this is an acceptable set of constraints from the point of view of the underlying gerbe as well as the universal gerbe depends on whether it forms, together with the BRST operator q, a closed algebra. This can be verified by calculating their covariant derivatives.

Theorem 5.

$$\delta^{(1)}_{\mu_i}\mathcal{C}_{ij} = \mathcal{B}_{ij} + [\iota_{V_{ij}}, \mathcal{C}_{ij}]$$
$$\delta^{(1)}_{\mu_i}\mathcal{C}_{ijk} = \mathcal{B}_{ijk} + [\mathcal{C}_{ij}, V_{ij}] + [{}^{\lambda_{ij}}\,\mathcal{C}_{jk}, V_{ij} + \lambda_{ij}V_{jk}] + [{}^{\lambda_{ij}\lambda_{jk}}\,\mathcal{C}_{ki}, \partial_\lambda V_{ij}]$$
$$\delta^{(2)}_{\mu_i}\mathcal{B}_{ij} = [\iota_{V_{ij}}, \mathcal{B}_{ij}] + [\iota_{A_i} + {}^{\lambda_{ij}}\,F_j, \mathcal{C}_{ij}]$$
$$\begin{aligned}\delta^{(2)}_{\mu_i}\mathcal{B}_{ijk} = {} & [{}^{\iota_{g_{ijk}}}\,F_i, \mathcal{C}_{ijk}] + [\lambda_{ij}\Delta_{jk} + \lambda_{ij}\lambda_{jk}\Delta_{ki}, \mathcal{C}_{ij}] + [\lambda_{ij}\lambda_{jk}\Delta_{ki}, {}^{\lambda_{ij}}\,\mathcal{C}_{jk}] \\ & + [V_{ij} + \lambda_{ij}V_{jk}, \mathcal{B}_{ijk}] - [\lambda_{ij}V_{jk}, \mathcal{B}_{ij}] + [\lambda_{ij}\lambda_{jk}V_{ki}, {}^{\lambda_{ij}\lambda_{jk}}\,\mathcal{B}_{ki}].\end{aligned}$$

From these results it is now possible to read off the actual constraint algebra. This is because the combinatorial differential includes the BRST differential q

$$\mathrm{q}_c\mathcal{C}_{ij} = \delta^{(1)}_{\mu_i}\mathcal{C}_{ij} - \mathrm{d}_{m_i}\mathcal{C}_{ij} - \frac{1}{2}[\mathcal{C}_{ij}, \mathcal{C}_{ij}] \tag{155}$$

$$\mathrm{q}_c\mathcal{C}_{ijk} = \delta^{(1)}_{\mu_i}\mathcal{C}_{ijk} - \mathrm{d}_{m_i}\mathcal{C}_{ijk} - \frac{1}{2}[\mathcal{C}_{ijk}, \mathcal{C}_{ijk}] \tag{156}$$

$$\mathrm{q}_c\mathcal{B}_{ij} = \delta^{(2)}_{\mu_i}\mathcal{B}_{ij} - \mathrm{d}_{m_i}\mathcal{B}_{ij} \tag{157}$$

$$\mathrm{q}_c\mathcal{B}_{ijk} = \delta^{(2)}_{\mu_i}\mathcal{B}_{ijk} - \mathrm{d}_{m_i}\mathcal{B}_{ijk}. \tag{158}$$

For instance,

$$\mathrm{q}_c\mathcal{C}_{ij} = \mathcal{B}_{ij} + [\iota_{V_{ij}}, \mathcal{C}_{ij}] - \mathrm{d}_{m_i}\mathcal{C}_{ij} - \frac{1}{2}[\mathcal{C}_{ij}, \mathcal{C}_{ij}]. \tag{159}$$

This can be decomposed order by order in ghost number

$$0 = {}^{\lambda_{ij}}\,\nu_j - \nu_i + \iota_{\delta_{ij}} - \mathrm{d}_{m_i-\gamma_{ij}}\mathcal{C}^0_{ij} - \frac{1}{2}[\mathcal{C}^0_{ij}, \mathcal{C}^0_{ij}] \tag{160}$$

$$\mathrm{q}_c\mathcal{C}^0_{ij} = \mathcal{B}^1_{ij} + [\iota_{a_{ij}}, \mathcal{C}^0_{ij}] - \mathrm{d}_{m_i-\gamma_{ij}}\mathcal{C}^1_{ij} - [\mathcal{C}^0_{ij}, \mathcal{C}^1_{ij}] \tag{161}$$

$$\mathrm{q}_c\mathcal{C}^1_{ij} = \mathcal{B}^2_{ij} + [\iota_{a_{ij}}, \mathcal{C}^1_{ij}] - \frac{1}{2}[\mathcal{C}^1_{ij}, \mathcal{C}^1_{ij}]. \tag{162}$$

The first of these equations can be checked independently by using the definitions of ν_i and δ_{ij} in terms of connection data. On-shell it reduces to the cocycle condition (39). The right-hand sides of the rest of the equations vanish on-shell, and the algebra closes.

There is one final twist to the constraint algebra: it is still reducible. This is because one can show again by direct calculation that the following relationships between the constraints apply:

$$\partial_\lambda\mathcal{C}_{ij} = \iota_{\mathcal{C}_{ijk}} \tag{163}$$

$$\partial_\lambda\mathcal{B}_{ij} = \iota_{\mathcal{B}_{ijk}}. \tag{164}$$

This means that we must, effectively, include these two equations in the constraint algebra as further constraints. We do this in the next section. In that analysis we shall need the following similar

Lemma 6. *The BRST transformations of the constraints are consistent on triple intersections in the sense* $\partial_\lambda \mathrm{q}\mathcal{C}_{ij} = \iota_{\mathrm{q}\mathcal{C}_{ijk}}$.

5.3. Constraints in the BRST cohomology

To trivialise the constraints $\mathcal{C}_{ij}, \mathcal{C}_{ijk}, \mathcal{B}_{ij}$, and $\mathcal{B}_{ijk}$ in BRST cohomology, we need to introduce two cohomologically trivial pairs of fields (Λ_{ij}, K_{ij}) and (Λ_{ijk}, K_{ijk}). Expanded in ghost number, the fields are

$$\Lambda_{ij} := \Lambda^{-1}_{ij} + \Lambda^0_{ij} \tag{165}$$

$$K_{ij} := K^0_{ij} + K^1_{ij}. \tag{166}$$

Table 4
Lagrange multiplies for imposing constraints

	Ghost#	0-form	1-form	2-form
G	-1		Λ^{-1}_{ijk}	
	0	Λ^{0}_{ijk}	K^{0}_{ijk}	
	1	K^{1}_{ijk}		
$\mathrm{Aut}(G)$	-2		M^{-2}_{ijk}	
	-1	M^{-1}_{ijk}	$\Lambda^{-1}_{ij}, N^{-1}_{ijk}$	
	0	$\Lambda^{0}_{ij}, N^{0}_{ijk}$	K^{0}_{ij}	
	1	K^{1}_{ij}		

We can now define their BRST transformations as

$$\mathrm{q}\Lambda_{ij} := \mathcal{C}_{ij} - K_{ij} \tag{167}$$
$$\mathrm{q}K_{ij} := \mathrm{q}_c\mathcal{C}_{ij} \tag{168}$$

and similarly for Λ_{ijk}, K_{ijk}. Here $\mathrm{q}\mathcal{C}_{ij}$ is a known expression, and reduces on-shell to the constraints $\mathrm{q}\mathcal{C}_{ij} \approx \mathcal{B}^1_{ij} + \mathcal{B}^2_{ij}$. The lowest order term $\mathcal{B}^0_{ij}$ does not, and should not, appear, as it is algebraically trivial. The BRST operator q is still nilpotent, and the constraints $\mathcal{C}_{ij}$ and $\mathcal{B}_{ij}$ are exact in the cohomology of q. (Note the absence of the ghost field c here. Any attempt at making (Λ_{ij}, K_{ij}) transform covariantly under q would lead to an accumulation of $\varphi_i + \iota_{\phi_i}$ terms that could not be cancelled.)

The reducibility relations observed in (163) and (164) can now be taken care of by introducing the ghost-for-ghost fields (M_{ijk}, N_{ijk}) and defining

$$\mathrm{q}M_{ijk} = \partial_\lambda \Lambda_{ij} - \iota_{\Lambda_{ijk}} + N_{ijk} \tag{169}$$
$$\mathrm{q}N_{ijk} = \partial_\lambda K_{ij} - \iota_{K_{ijk}}. \tag{170}$$

We have summarised the new fields required for trivialising the constraints in Table 4. This table includes fields of so negative ghost number that their total degree as universal forms is negative, indeed -1 for the components of M_{ijk}. From the field theory point of view this is of no consequence. From the point of view of the universal gerbe the situation is slightly odd, however, and may suggest that we should see the Čech degree as a part of the grading. Then the total degree of M_{ijk} is zero and N_{ijk} is one. Similarly the degree of Λ_{ij}, K_{ij} is then one and Λ_{ijk}, K_{ijk} is two, and the Čech differential ∂_λ raises the degree by one.

The constraint algebra closes now, the full BRST operator q is identically nilpotent, takes into account all the reducibility relations, and its cohomology is supported on the constraint surface

$$\mathcal{C}_{ij} \approx \mathcal{C}_{ijk} \approx \mathcal{B}_{ij} \approx \mathcal{B}_{ijk} \approx 0. \tag{171}$$

Assuming that we have the traces tr_i, Tr_i and the Hodge star $*$ of a Euclidean metric at our disposal (cf. Section 7.1), a suitable gauge fermion that imposes these constraints in a path integral is

$$\Psi = \mathrm{Tr}_i \Lambda_{ij} \wedge *K_{ij} + \mathrm{tr}_i \Lambda_{ijk} \wedge *K_{ijk} + \mathrm{Tr}_i M_{ijk} \wedge *N_{ijk}. \tag{172}$$

Integrating out N one gets a Gaussian suppression for the norm of $\partial_\lambda \Lambda_{ij} - \iota_{\Lambda_{ijk}}$, and the path integral over M forces $\partial_\lambda K_{ij} - \iota_{K_{ijk}} = 0$. Λ and K act as Lagrange multipliers for $\mathcal{C}$ and $\mathcal{B}$ respectively.

6. Comparison

We have presented in Sections 3 and 5 two very similar constructions that nevertheless differ in certain details. To show that they are mathematically equivalent one would have to demonstrate that the cohomologies of Q and q are isomorphic. Of course, as one of the operators, Q, is not nilpotent this cannot be done directly.

The field space where the nilpotent operator q acts, is larger than the one where Q does. The operators can, therefore, be compared only in a locus where the additional fields a_{ij} and b_{ij} are somehow eliminated. In a classical physical theory this could be done by imposing equations of motion; unfortunately, in want of an action principle, we do not have enough information to do so, nor should we indeed impose classical equations of motion on fields which we plan to quantise.

What we really need to show, in fact, is that any path integral with a q-invariant measure and a q-invariant integrand localises in the new fields a_{ij} and b_{ij} and that the effective BRST operator q acts in this locus as Q. *This means that the quantum cohomology of* q *is cohomology of the fully decomposed gerbe* with fixed cocycle data.[7] This localisation does indeed happen, and the loci where the path integral localises are the fixed point loci of q.

6.1. Grading

Let us start by eliminating the most obvious difference, namely that of grading. In Section 3 the Lie-bracket of two fields x and y (in a fixed representation) with bigradings (p, q) and (p', q') was defined

$$[x, y] = \begin{cases} xy - (-)^{pp'+qq'} yx & \text{in Section 3} \\ xy - (-)^{(p+q)(p'+q')} yx & \text{in Section 5.} \end{cases} \tag{173}$$

Also the two BRST operators behaved differently in the presence of an exterior derivative: for the former we have $\mathrm{Qd} = \mathrm{dQ}$, whereas for the latter $\mathrm{qd} = -\mathrm{dq}$.

We can map the constructions one to the other by

(a) Mapping every quadratic object xy in the BRST transformation rules

$$xy \mapsto (-)^{pq'} xy, \tag{174}$$

where p is the form degree of x and q' is ghost number of y.

(b) Redefining fields

$$(c_i, \varphi_i, \phi_i, \rho_i, \sigma_i) \mapsto (-c_i, -\varphi_i, -\phi_i, -\rho_i, -\sigma_i). \tag{175}$$

This mapping is well-defined in the sense that the result does not depend on the order in which the bilinears are written down. It also leaves the curvature triple unchanged. There are changes in the new ghost constraints (138) and (152)–(154). Applying these rules we get the nilpotent extension $\bar{\mathrm{q}}$ of Q

$$\bar{\mathrm{q}} m_i = \pi_i + \iota_{E_i} - \mathrm{d}_{m_i} c_i \tag{176}$$
$$\bar{\mathrm{q}}_c \gamma_{ij} = \eta_{ij} + E_i - \lambda_{ij}(E_j) + \mathrm{d}_{m_i} a_{ij} - [\gamma_{ij}, a_{ij}] \tag{177}$$
$$\bar{\mathrm{q}}_c B_i = \alpha_i + \mathrm{d}_{m_i} E_i \tag{178}$$
$$\bar{\mathrm{q}}_c \pi_i = \iota_{\rho_i} + \mathrm{d}_{m_i} \varphi_i \tag{179}$$
$$\bar{\mathrm{q}}_c E_i = -\rho_i + \mathrm{d}_{m_i} \phi_i \tag{180}$$
$$\bar{\mathrm{q}} c_i = \varphi_i + \iota_{\phi_i} + \frac{1}{2}[c_i, c_i] \tag{181}$$
$$\begin{aligned} \bar{\mathrm{q}}_c \eta_{ij} = {} & -\mathrm{d}_{m_i} b_{ij} + \rho_i - \lambda_{ij}(\rho_j) + [\iota_{\eta_{ij}} - \pi_i, a_{ij}] - [\varphi_i + \iota_{b_{ij}}, \gamma_{ij}] \\ & + [\mathcal{C}^0_{ij}, \lambda_{ij}(\phi_j)] - [\mathcal{C}^1_{ij}, \lambda_{ij}(E_j)] \end{aligned} \tag{182}$$
$$\bar{\mathrm{q}}_c \alpha_i = \mathrm{d}_{m_i} \rho_i - [\nu_i, \phi_i] - [\pi_i, E_i] - [\varphi_i, B_i] \tag{183}$$
$$\bar{\mathrm{q}}_c \varphi_i = -\iota_{\sigma_i} \tag{184}$$
$$\bar{\mathrm{q}}_c \phi_i = \sigma_i \tag{185}$$
$$\bar{\mathrm{q}}_c \rho_i = \mathrm{d}_{m_i} \sigma_i + [\pi_i, \phi_i] + [\varphi_i, E_i] \tag{186}$$
$$\bar{\mathrm{q}}_c \sigma_i = -[\varphi_i, \phi_i] \tag{187}$$

[7] This is cohomology of the fields living on the non-Abelian gerbe, not the cohomology group $H^1(X, \mathbf{G})$ of Section 2.1.

$$\bar{\mathrm{q}}_c a_{ij} = b_{ij} - \phi_i + \lambda_{ij}(\phi_j) + \frac{1}{2}[a_{ij}, a_{ij}] \tag{188}$$

$$\bar{\mathrm{q}}_c b_{ij} = \sigma_i - \lambda_{ij}(\sigma_j) - [\varphi_i + \iota_{b_{ij}}, a_{ij}] + [\mathcal{C}^1_{ij}, \lambda_{ij}(\phi_j)] \tag{189}$$

$$\bar{\mathrm{q}}\Lambda_{ij} = \mathcal{C}_{ij} - K_{ij} \tag{190}$$

$$\bar{\mathrm{q}}K_{ij} = \bar{\mathrm{q}}_c\mathcal{C}_{ij} \tag{191}$$

$$\bar{\mathrm{q}}\Lambda_{ijk} = \mathcal{C}_{ijk} - K_{ijk} \tag{192}$$

$$\bar{\mathrm{q}}K_{ijk} = \bar{\mathrm{q}}_c\mathcal{C}_{ijk} \tag{193}$$

$$\bar{\mathrm{q}}M_{ijk} = \partial_\lambda\Lambda_{ij} - \iota_{\Lambda_{ijk}} + N_{ijk} \tag{194}$$

$$\bar{\mathrm{q}}N_{ijk} = \partial_\lambda K_{ij} - \iota_{K_{ijk}}. \tag{195}$$

This differs from Q in the definitions of $\bar{\mathrm{q}}\gamma_{ij}$ and $\bar{\mathrm{q}}\eta_{ij}$, and in that it involves the auxiliary fields a_{ij} and b_{ij}.

6.2. *On-shell algebra*

The discussion of Section 5.3 guarantees that we can make the path integral localise on subsets of the field space where the constraints vanish. On that surface we can define an effective BRST operator $\hat{\mathrm{q}}$ that is formed from $\bar{\mathrm{q}}$ by simply dropping the constraints that appear explicitly in the definitions of $\bar{\mathrm{q}}\eta_{ij}$ and $\bar{\mathrm{q}}b_{ij}$

$$\hat{\mathrm{q}}_c\eta_{ij} = -\mathrm{d}_{m_i}b_{ij} + \rho_i - \lambda_{ij}(\rho_j) - [\iota_{\eta_{ij}} + \pi_i, a_{ij}] - [\varphi_i + \iota_{b_{ij}}, \gamma_{ij}] \tag{196}$$

$$\hat{\mathrm{q}}_c b_{ij} = \sigma_i - \lambda_{ij}(\sigma_j) - [\varphi_i + \iota_{b_{ij}}, a_{ij}], \tag{197}$$

and $\hat{\mathrm{q}}x := \bar{\mathrm{q}}x$ for any other field x. This operator continues to be nilpotent on the constraint surface, as can be seen using

Lemma 7.

$$\hat{\mathrm{q}}^2\gamma_{ij} = -[\mathcal{C}^0, \lambda_{ij}(\phi_j)] + [\mathcal{C}^1, \lambda_{ij}(E_j)] \tag{198}$$

$$\hat{\mathrm{q}}^2\eta_{ij} = -[\mathcal{B}^1, \lambda_{ij}(\phi_j)] + [\mathcal{B}^2, \lambda_{ij}(E_j)] + [\mathcal{C}^0, \lambda_{ij}(\sigma_j)] + [\mathcal{C}^1, \lambda_{ij}(\rho_j)] \tag{199}$$

$$\hat{\mathrm{q}}^2 a_{ij} = -[\mathcal{C}^1, \lambda_{ij}(\phi_j)] \tag{200}$$

$$\hat{\mathrm{q}}^2 b_{ij} = -[\mathcal{B}^2, \lambda_{ij}(\phi_j)] + [\mathcal{C}^1, \lambda_{ij}(\sigma_j)], \tag{201}$$

and $\hat{\mathrm{q}}^2 x = 0$ *for all other fields.*

In comparing Theorem 2 and Lemma 7 we notice that the terms involving ${}^{\lambda_{ij}}\varphi_j - \varphi_i$ and ${}^{\lambda_{ij}}c_j - c_i$ in Theorem 2 have been completed to the constraints $\mathcal{B}^2_{ij}$ and $\mathcal{C}^1_{ij}$ in Lemma 7, respectively. (Other differences have to do with the consistent treatment and elimination of the new fields a_{ij} and b_{ij}.)

As the original symmetries of the gerbe made use of constraints as cocycle conditions, we should compare $\hat{\mathrm{q}}$ (rather than the nilpotent $\bar{\mathrm{q}}$) with Q. What the above discussion shows is that, on the constraint surface, we can indeed turn $\bar{\mathrm{q}}$ consistently into a non-nilpotent on-shell operator $\hat{\mathrm{q}}$ whose action generalises, in a certain way, that of Q.

6.3. *Eliminating auxiliaries*

Having dealt with the constraints that appear explicitly in the definition of $\bar{\mathrm{q}}$, we are ready to investigate the rôle played by the auxiliary fields a_{ij} and b_{ij}. For this we need the following

Lemma 8. *Let the odd vector field S on V be a symmetry of both the measure* μ *and the function* I*. Then the integral* $\int \mu I$ *is supported only at the fixed point loci of* S *in* V*.*

Proof. The well-known argument [33] is as follows: let $S = \partial/\partial\theta$ be an anticommuting vector field on V, and θ the local anticommuting coordinate along which S generates translations. Such a coordinate exists wherever the action of S is free. Let S act freely on $U \subset V$, so that $\mu = \mu' \wedge \mathrm{d}\theta$ and $SI = 0$. Then

$$\int_U \mu I = \int_{U'} \mu' \frac{\partial}{\partial\theta} I = 0 \tag{202}$$

by the properties of the Berezin integral. Hence the only nontrivial contributions can arise from the fixed point set of S in V. □

Requiring that q should act consistently on a_{ij}, i.e. $\mathrm{q}a_{ij} = 0$, fixes b_{ij} as a functional of other fields in the theory. At this locus we have

$$a_{ij} = \tilde{a}_{ij} \tag{203}$$

$$b_{ij} = \phi_i - \lambda_{ij}(\phi_j) - \frac{1}{2}[a_{ij}, a_{ij}] - [c_i, a_{ij}], \tag{204}$$

where $\tilde{a}_{ij}$ is a fixed background field $\mathrm{q}\tilde{a}_{ij} = 0$. Possible values include, but are not restricted to, $\tilde{a}_{ij} = 0$. One can check

$$\bar{\mathrm{q}}\left(\phi_i - \lambda_{ij}(\phi_j) - \frac{1}{2}[a_{ij}, a_{ij}] - [c_i, a_{ij}]\right) = \bar{\mathrm{q}} b_{ij}. \tag{205}$$

This means that we can use (204) as an algebraic identity. By Lemma 7, any $\bar{\mathrm{q}}$-invariant path integral then localises to the values of a_{ij} and b_{ij} given in (203) and (204).

We may now make use of the above values of a_{ij} and b_{ij}, and compare the transformation rules on-shell for Q and $\hat{\mathrm{q}}$. Those that are functionally different are

$$\hat{\mathrm{q}}\gamma_{ij} = \mathrm{Q}\gamma_{ij} + \mathrm{d}_{m_i - \iota_{\gamma_{ij}}} \tilde{a}_{ij} \tag{206}$$

$$\hat{\mathrm{q}}\eta_{ij} = \mathrm{Q}\eta_{ij} + [\iota_{\eta_{ij}} - \pi_i + \mathrm{d}_{m_i - \iota_{\gamma_{ij}}} \tilde{a}_{ij}, \tilde{a}_{ij}] + \mathrm{d}_{m_i - \iota_{\gamma_{ij}}} [c_i, \tilde{a}_{ij}]. \tag{207}$$

When $\tilde{a}_{ij} = 0$ we see that $\hat{\mathrm{q}}$ and Q agree.

It is not quite clear from this analysis what rôle the other vacua with $\tilde{a}_{ij} \neq 0$ play. One possibility is that one may be able to localise a_{ij} at $a_{ij} = 0$ in the path integral by suitable gauge fixing. If this is the case, then the constraint $\mathcal{C}^1_{ij}$ will force the local Faddeev–Popov ghosts c_i to form a globally well defined scalar field. This would mean that local gauge transformations on different charts must be globally consistent: the gauge is the same everywhere.

7. Notes on observables

Observables $\mathcal{O}$ are BRST-closed $\mathrm{q}\mathcal{O} = 0$ functionals on the field space. The vacuum expectation values of BRST-exact functionals vanish. Physical states belong to the cohomology of q. Determining that cohomology is a fundamental problem in Quantum Field Theory.

In this section we look for observables for a fully decomposed non-Abelian gerbe. It turns out that the standard field theory methods do not quite suffice, and the outer part of the automorphism group plays a crucial rôle.

7.1. Local traces

Due to the freedom to choose the frame in the local gauge symmetry, observables $\mathcal{O}$ should first of all not carry bare indices in representations of G or $\mathrm{Aut}\, G$. This is because no covariant quantity x is BRST-closed: $\mathrm{q}x = -[c, x] + \cdots$ does not vanish identically.

Given on each chart $\mathcal{U}_i$ a finite dimensional linear representations of G and $\mathrm{Aut}\, G$ – or indeed of the local groups G_i and $\mathrm{Aut}\, G_i$, to be more precise – we have the traces

$$\mathrm{tr}_i : G_i \longrightarrow \mathbb{R} \tag{208}$$

$$\mathrm{Tr}_i : \mathrm{Aut}\, G_i \longrightarrow \mathbb{R} \tag{209}$$

at our disposal. Traces are not invariant in outer automorphisms, so this does not provide, directly, local invariants for a given field configuration. Since we are at liberty to define each trace locally as we please, we may nevertheless choose them to be compatible in the following sense:

$$\mathrm{tr}_i \lambda_{ij}(x_j) = \mathrm{tr}_j x_j, \tag{210}$$

and similarly for Tr_i. It would *not* have been possible to assume them to be invariant under arbitrary automorphisms — the λ_{ij} are special. The cyclic property of the finite dimensional trace guarantees that these choices can be done in a globally consistent way

$$\mathrm{tr}_i \lambda_{ij}\lambda_{jk}\lambda_{ki}(x_i) = \mathrm{tr}_i \left(g_{ijk}\; x_i\; g_{ijk}^{-1} \right) = \mathrm{tr}_i x_i, \tag{211}$$

and similarly for Tr_i. In traditional Quantum Field Theory typical observables are indeed "invariant[8]" polynomials, i.e. combinations of traces of covariant operators, such as Chern classes.

In the pure Yang–Mills case the BRST operator q reduces to the covariant exterior derivative q^H on the bundle $\mathcal{A} \longrightarrow \mathcal{A}/\mathcal{G}$ whose fibre is the gauge group $\mathcal{G} = \underline{\Omega}^0(X, G)$. The curvature of this differential is one of the scalar fields in the theory, and hence nontrivial. Nevertheless, operated on invariant polynomials on the base space $\mathcal{A}/\mathcal{G}$, q^H is nilpotent – thanks to the fact that traces of commutators vanish $\phi \in \mathcal{G}$

$$(\mathrm{q}^H)^2 \mathrm{tr} x = \mathrm{tr}[\phi, x] = 0. \tag{212}$$

In the context of a non-Abelian gerbe, this does not happen, for several reasons:

(i) Invariance does not imply well-definedness on intersections, as even the curvature triple may jump there, cf. (36), (39) and (40).

$$^{\lambda_{ij}} F_j - F_i \approx -\iota_{\Delta_{ij}} \tag{213}$$

$$\partial_{\lambda_{ij}}(\Delta_{ij}) \approx [F_i, g_{ijk}] \tag{214}$$

$$\lambda_{ij}(\Omega_j) - \Omega_i \approx \mathrm{d}_{\mu_i - \iota_{V_{ij}}}(\Delta_{ij}) + [V_{ij}, F_i]. \tag{215}$$

(ii) The curvature of q^H is given locally on $\mathcal{U}_i$ by $\varphi_i + \iota_{\phi_i}$, but since φ_i is not an inner automorphism there is no guarantee that the square $(\mathrm{q}^H)^2$ should vanish on traces

$$(\mathrm{q}^H)^2 \mathrm{tr}_i x_i = \mathrm{tr}_i[\varphi_i + \iota_{\phi_i}, x_i] = \mathrm{tr}_i[\varphi_i, x_i] \not\equiv 0. \tag{216}$$

(iii) Gauge structure is not global; covariant derivatives q^H on different charts do not glue together consistently on $\mathcal{U}_{ij}$, cf. Section 5.1.1.

On the other hand, *it is precisely these complications that make it possible for outer automorphisms to appear in BRST cohomology, and to make contact with the cohomology of non-Abelian gerbes in* Section 2.1. Despite these difficulties, traces have the following two useful properties:

Lemma 9. *Cyclicity of the finite dimensional trace implies*

$$\mathrm{tr}_i \mathrm{d}_{m_i} \lambda_{ij} X_j \approx \mathrm{tr}_i \lambda_{ij} \mathrm{d}_{m_j} X_j. \tag{217}$$

When the connection one-form is inner, i.e. $m_i = \iota_{n_i}$ *for some* $n_i \in \Omega^1(\mathcal{U}_i, G)$,

$$\mathrm{d}\mathrm{tr}_i X_i = \mathrm{tr}_i \mathrm{d}_{\iota_{n_i}} X_i. \tag{218}$$

Proof. The first point follows upon using the constraint $\mathcal{C}_{ij} \approx 0$ and the fact group-valued one-form $\mathrm{tr}_i[\gamma_{ij}, X_i] = 0$. □

For general Aut G-valued forms m (218) is not true as $\mathrm{tr}_i[m, X]$ does not necessarily vanish. Traces of commutators vanish only when the automorphism m happens to be inner $m \in \mathrm{im}\iota$ and its form degree positive.

[8] Invariant under *inner* automorphisms.

7.2. Fake curvature and Donaldson–Witten invariants

The natural generalisation of the second Chern class that appeared in Donaldson–Witten theory is to replace $\kappa(m_i)$ with the fake curvature ν_i and use the local trace

$$\frac{1}{2}\mathrm{Tr}_i F_i \wedge F_i = \mathrm{Tr}_i \left(\frac{1}{2}\nu_i \wedge \nu_i + \pi_i \wedge \nu_i + \varphi_i \wedge \nu_i + \frac{1}{2}\pi_i \wedge \pi_i + \pi_i \wedge \varphi_i + \varphi_i \wedge \varphi_i\right). \tag{219}$$

This can be thought of as a local deformation of the Donaldson–Witten invariants by ι_{B_i}. Unlike the Chern class used in Donaldson–Witten theory, (219) does not determine an element in $H^*(X, \mathbb{R})$, however:

- It is not globally defined, unless $\iota_{\Delta_{ij}}$ vanishes. This is because

$$^{\lambda_{ij}} F_j = F_i - \iota_{\Delta_{ij}}. \tag{220}$$

- It is not closed, unless ι_{Ω_i} vanishes

$$(\mathrm{d} + \mathrm{q})\frac{1}{2}(\mathrm{Tr}_i F_i \wedge F_i) = -\mathrm{Tr}_i(\iota_{\Omega_i} \wedge F_i). \tag{221}$$

Note that the right-hand side in (221) is a globally on X defined differential form precisely when (219) is. But then (221) is cohomologically trivial and does not lead to interesting observables. The Donaldson–Witten invariants are produced, in fact, only in the essentially Abelian case where

$$\iota_{\Omega_i} = \iota_{\Delta_{ij}} = 0. \tag{222}$$

This can of course be arranged by assuming

$$\iota_{A_i} = \delta^{(1)}_{\mu_i}(-\iota_{V_{ij}}) = 0. \tag{223}$$

7.3. Abelian cases

In this section we shall investigate the cohomology of the trace part of a non-Abelian gerbe: this leads to the Abelian gerbe with structure of [28].[9]

Suppose m_i is in the image of ι so that Lemma 9 holds. Let us consider the trace parts of the rest of the connection data

$$\bar{B}_i := \mathrm{tr}_i B_i \tag{224}$$

$$\bar{A}_{ij} := \mathrm{tr}_i \gamma_{ij} \tag{225}$$

$$\bar{g}_{ijk} := \det_i g_{ijk}. \tag{226}$$

The corresponding three-form $\bar{\omega}_i := \mathrm{d}\bar{B} = \mathrm{tr}_i \mathrm{d}_{m_i} B_i$ is now by (38) closed, and satisfies by (40)

$$\bar{\omega}_i = \bar{\omega}_j + \mathrm{d}\bar{\delta}_{ij} \tag{227}$$

where again $\bar{\delta}_{ij} = \mathrm{tr}_i \delta_{ij}$. As long as there is no more information about $\bar{\delta}_{ij}$, the local three-forms do not patch together in any useful way.

Suppose further that $\bar{\delta}_{ij} = 0$. Then $\bar{\omega}_i$ extends to a well-defined global differential form $\bar{\omega} \in \Omega^3(X, \mathbb{R})$. (Note that this implies $\mathrm{q}\bar{\delta}_{ij} = 0$, which leads to further conditions between ghost fields $\bar{\alpha}_j - \bar{\alpha}_i + \mathrm{d}\bar{\eta}_{ij} \equiv 0$ modulo traces of commutators.) The cocycle conditions (28), (34) and (35) take the form

$$\bar{B}_j - \bar{B}_i + \mathrm{d}\bar{A}_{ij} = 0 \tag{228}$$

$$\bar{A}_{ij} + \bar{A}_{jk} + \bar{A}_{ki} - \mathrm{d}\ln \bar{g}_{ijk} = 0 \tag{229}$$

$$\bar{g}_{jkl}\bar{g}_{ijl}\bar{g}^{-1}_{ijk}\bar{g}^{-1}_{ikl} = 1, \tag{230}$$

[9] Note that when the λ_{ij} part of the cocycle data is trivial, the gerbe is called Abelian in [3]. Indeed, this implies $\iota_{g_{ijk}} = 0$. A fully decomposed Abelian gerbe is discussed in detail in Section 7.3 of Ref. [2].

where we used $\mathrm{tr}_i \delta^{(0)} g_{ijk}^{-1} = -\mathrm{d} \ln \bar{g}_{ijk}$. This defines a representative of a class $[\bar{B}_i, \bar{A}_{ij}, \bar{g}_{ijk}^{-1}]$ in the standard Čech–de Rham cohomology or, in other words, an Abelian gerbe with connection and curving [28].

It is interesting to find the part of the symmetries of the non-Abelian gerbe that correspond to the standard action of a Čech–de Rham one-cochain on the above two-cocycle.

The symmetries of the non-Abelian gerbe involve among other generators $\bar{E}_i := \mathrm{tr}_i E_i$ and $\bar{a}_{ij} := \mathrm{tr}_i a_{ij}$. As the one-cochain involves real fields and not ghosts, we need to consider E_i, a_{ij} as elements of $\Omega^1(\mathcal{U}_i, \mathrm{Lie}\, G)$ and $\Omega^0(\mathcal{U}_{ij}, \mathrm{Lie}\, G)$. As we have argued in Section 6.1, this change of grading forces us to change the sign in front of all exterior derivatives $\mathrm{d} \longrightarrow -\mathrm{d}$. With this understanding, (127) and (129) lead to

$$\bar{B}_i' = \bar{B}_i + \mathrm{d}\bar{E}_i \tag{231}$$

$$\bar{A}_{ij}' = \bar{A}_{ij} - \bar{E}_j + \bar{E}_i + \mathrm{d}\bar{a}_{ij} \tag{232}$$

$$\ln \bar{g}_{jkl}' = \ln \bar{g}_{ijk} + \bar{a}_{ij} + \bar{a}_{jk} + \bar{a}_{ki}. \tag{233}$$

The two first rules can be read off, of course, directly from the definition of $\bar{\mathrm{q}}$ as well.

The last transformation rule (233) may appear surprising, however, given that $\bar{\mathrm{q}}$ annihilates all cocycle data (λ_{ij}, g_{ijk}); yet it is required to keep (229) invariant. It can be derived as follows: if we want to compare the values of a group-valued function $g(x)$ at two different points on $x, y \in X$ we have to parallel transport the group element from one point to the other *covariantly* to be able to perform the comparison. The difference is then given precisely by the combinatorial derivative $m(x, y)(g(y))g(x)^{-1} = \tilde{\delta}_m^{(0)} g(x, y)$.

The same is true of comparing the values of the group element in different points x, ξ on the universal gerbe $\mathfrak{G}$ on the same orbit of the action of the symmetry group $\xi \in \hat{\mathcal{G}} \cdot x$. As the group element g_{ijk} is constant $\bar{\mathrm{q}} g_{ijk}(x, \xi) = \tilde{\delta}^{(0)} g(x, \xi) = 1$ in these transformations, we have $g_{ijk}(\xi) = g_{ijk}(x)$. Nevertheless, the frame in $\mathfrak{G}$ changes along the way due to the presence of the curvature of the connection $\mu_i(x, \xi) = -c_i(x)$ so that

$$g_{ijk}' g_{ijk}^{-1} = \mu_i(x, \xi)(g_{ijk}(\xi)) g_{ijk}^{-1}(x) \tag{234}$$

$$= \tilde{\delta}_\mu^{(0)}(g_{ijk})(x, \xi) \tag{235}$$

$$= -c_i(g_{ijk}) g_{ijk}^{-1}. \tag{236}$$

The part of the symmetry group that is responsible for this change is clearly the group of local gauge transformations $\mathcal{G}_i \subset \hat{\mathcal{G}}$. As discussed in Section 4.2, the (locally defined) covariant exterior derivative $\bar{\mathrm{q}}^{\mathrm{H}}$ on the base space $\mathfrak{G}/\mathcal{G}_i$ can be obtained from $\bar{\mathrm{q}}$ formally by setting $c_i = 0$. (Note that the fields η_{ij} and b_{ij} that caused trouble in Section 5.1.1 should here be set to trivial values.) This means that on this base space g_{ijk} is covariantly constant $\bar{\mathrm{q}}^{\mathrm{H}} g_{ijk} = 0$. The extra terms in (233) appear therefore as a consequence of eliminating the non-Abelian symmetry $\mathcal{G}_i \subset \hat{\mathcal{G}}$, and restricting to basic cohomology on $\mathfrak{G}/\mathcal{G}_i$.

The Abelian part transforms then

$$\ln \bar{g}_{ijk}' = \ln \det -c_i(g_{ijk}) \tag{237}$$

$$= \ln \bar{g}_{ijk} + \ln \det[-c_i, g_{ijk}] \tag{238}$$

$$= \ln \bar{g}_{ijk} + \mathrm{tr}_i[-c_i, g_{ijk}] \tag{239}$$

$$\approx \ln \bar{g}_{ijk}' + \partial_\lambda \bar{a}_{ij}. \tag{240}$$

We have used at (239) the fact that c_i is really a one-form, and at (240) the constraint $\mathcal{C}_{ijk}^1 \approx 0$. Similarly, under $\bar{\phi}_i := \mathrm{tr}_i \phi_i$,

$$\bar{E}_i' = \bar{E}_i + \mathrm{d}\bar{\phi}_i \tag{241}$$

$$\bar{a}_{ij}' = \bar{a}_{ij} + \bar{\phi}_j - \bar{\phi}_i. \tag{242}$$

There are two obvious candidates for observables, but both fail to be BRST-closed unless we impose conditions on $\bar{\alpha}_i$ and $\bar{\eta}_{ij}$.

- The three-form ω_i. It fails to be closed by $\bar{\mathrm{q}}\bar{\omega}_i = \mathrm{d}\bar{\alpha}_i$.

- Given a triangulation with sides s, edges e and vertices v of a three-dimensional surface M, we can define the holonomy [28]

$$\mathrm{hol}_{[\bar{B}_i,\bar{A}_{ij},\bar{g}_{ijk}^{-1}]} = \sum_{s\subset M}\int_s \bar{B}_s + \sum_{e\subset\partial s}\int_e \bar{A}_{es} + \sum_{v\in\partial e}\ln\bar{g}_{ves}^{-1} \tag{243}$$

which transforms by the holonomy

$$\mathrm{hol}_{[\bar{E}_i,\bar{a}_{ij}]} = \sum_{e\subset\partial M}\int_e \bar{E}_e - \sum_{v\in\partial e}\bar{a}_{ve} \tag{244}$$

under Abelian symmetries, but picks up the extra piece $\mathrm{hol}_{[\bar{\alpha}_i,\bar{\eta}_{ij},1]}$ under non-Abelian symmetries. The former transformation vanishes on closed surfaces $\partial M = 0$ whereas the latter does not.

8. Discussion

We have proposed two equivalent constructions for a nilpotent BRST operator $\bar{\mathrm{q}}$ and q that both generate infinitesimal symmetries of a non-Abelian gerbe, though on differently-graded differential forms. For this it was crucial to arrange the cocycle conditions of [2] in two categories:

- The constraints that the gauge potentials in connection data satisfy;
- The Bianchi identities that the curvature triple satisfy on-shell.

This was possible, as the curvature triple turned out to be completely determined once the connection data was given.

This is exactly what is needed for defining a path integral measure in quantising the theory as well: the measure can now be easily written down by integrating over all free fields (connection data, affine data, Lagrange multipliers) and imposing the constraints with the help of a gauge fermion, such as (172). Having thus defined the measure, we are nevertheless still lacking a local invariant action principle that would lead to a finite path integral, and well-defined correlators for observables.

It would now be interesting to determine the BRST cohomology in terms of functionals composed of fields living on the gerbe. Standard methods in QFT do not seem to be able to catch the special features associated with the crossed module $G \longrightarrow \mathrm{Aut}\,G$ but tend to collapse it to an Abelian $\mathrm{Z}G$ gerbe. There are indeed three crucial differences to traditional Topological Quantum Field Theory:

- Traces of commutators such as $\mathrm{tr}_i[m_i, B_i]$ do not vanish, unless both operators are group-valued;
- Traces of differential forms are invariant polynomials only in the sub-sector of the theory where φ_i is in the image of ι, i.e. it is an inner automorphism;
- Locally invariant polynomials are not necessarily globally invariant, if they involve either η_{ij} or b_{ij}.

The BRST algebra we have found is not affected directly by any of these phenomena. However, it is precisely these features that are sensitive to the effects of the outer part of the automorphism group $\mathrm{Out}\,G$, and are likely to make it possible to recover some of the structure of the underlying cohomology of the gerbe $H^1(\mathbf{G})$.

In standard Yang–Mills theory, gauge invariant observables were easily identified as elements of the basic complex, and the BRST operator turned out to be the associated covariant derivative. This structure is repeated here only outside double intersections. On double intersections the action of the horizontal BRST operator, e.g. on η_{ij}, contains extra pieces that do not have the interpretation as a curvature. If one nevertheless restricts to configurations where the naïve curvature is fully covariant ${}^{\lambda_{ij}}(\varphi_j + \iota_{\phi_j}) = \varphi_i + \iota_{\phi_i}$, the mismatch vanishes.

Neither c_i, nor φ_i, nor ϕ_i can in general be assumed to extend to an everywhere well-defined object. To keep track of these mismatches in local gauge structure, we had to introduce the new fields a_{ij} and b_{ij} that were not present in the original fully decomposed gerbe. At the fixed point locus of the BRST operator it turned out that b_{ij} was essentially the failure of ϕ_i to extend to a global section, and that the constraint $\mathcal{B}^2_{ij} \approx 0$ then effectively guaranteed – again, only at the fixed point locus $a_{ij} = 0$ – that $\varphi_i + \iota_{\phi_i}$ should indeed transform covariantly from one chart to another with λ_{ij}. At this locus we can define basic functionals that are invariant under local gauge transformations (though only under inner automorphisms), and quotient out consistently the inner part of the local gauge groups $\mathcal{G}_i$.

The mismatch in c_i was measured in terms of $\iota_{a_{ij}}$. This field was required in Section 7.3 for realising the Abelian gerbe's symmetries consistently. At the fixed point locus we could choose any fixed background value $a_{ij} = \tilde{a}_{ij}$, though the trivial value $a_{ij} = 0$ was the one that reproduced the BRST operator of the non-Abelian gerbe. It remains an interesting problem to understand the significance of these other fixed point loci.

The use of combinatorial differential geometry simplified further certain standard operations in BRST quantisation. For instance, ghost number grading is easy to implement in terms of combinatorial differential geometry; this led to insights in the gauge structure that would otherwise be rather difficult to achieve. This became particularly obvious in the calculation of the constraint algebra, and in extracting the Abelian Čech–de Rham structure.

The present structure differs in fact from direct generalisations of the Čech–de Rham treatment of Abelian gerbes such as [34] for instance through the presence of δ_{ij}. Only setting this part of curvature to zero do we get the familiar relationship between a jump in the B_i-field and the exterior derivative of a one-form γ_{ij}. Furthermore, in the present considerations the analogue of the Čech coboundary operator ∂_λ did not change the grading or the degree of the fields on which it operated. This is in contrast with the Abelian case, where the connection and the curving of a gerbe fit in a Čech–de Rham cocycle where the Čech and the de Rham form degree are on equal footing. It was only in discussing the ghost number assignments of the Lagrange multipliers for the constraints that it seemed reasonable to take the Čech degree to contribute to the total grading.

Finally, it would be interesting to calculate the cohomology of the BRST operator and to compare it to the cohomology of the underlying gerbe. Also, a non-trivial action principle for path integral quantisation is still lacking. The results presented here will hopefully open doors for making use of these structures more directly in String and Quantum Field Theory, cf. [35]. Possible applications where the rôle of the automorphism group comes to its full right are situations where local perturbative descriptions of a quantum field theory differ globally by non-perturbative symmetry operations, e.g. in non-geometric backgrounds of String Theory.

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